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The Scott Topology. Part II[1]

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Summary. Mizar formalization of pp. 105–108 of [15] which continues [34]. We found a simplification for the proof of Corollary 1.15, in the last case, see the proof in the Mizar article for details.

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The terminology and notation used in this paper are introduced in the following articles: [30], [37], [10], [2], [25], [14], [29], [38], [8], [9], [35], [3], [1], [36], [27], [39], [13], [26], [31], [17], [28], [18], [12], [4], [16], [41], [19], [20], [33], [6], [32], [5], [11], [21], [7], [40], [23], [24], [22], and [34].

1. Preliminaries

The following propositions are true:

(1) Let X be a set and F be a finite family of subsets of X. Then there exists a finite family G of subsets of X such that $G \subseteq F$ and $\bigcup G = \bigcup F$ and for every subset g of X such that $g \in G$ holds $g \not\subseteq \bigcup(G \setminus \{g\})$.

(2) Let S be a 1-sorted structure and X be a subset of the carrier of S. Then $-X =$ the carrier of S if and only if X is empty.

(3) Let R be an antisymmetric transitive non empty relational structure with g.l.b.'s and x, y be elements of R. Then $\downarrow(x \sqcap y) = \downarrow x \cap \downarrow y$.

(4) Let R be an antisymmetric transitive non empty relational structure with l.u.b.'s and x, y be elements of R. Then $\uparrow(x \sqcup y) = \uparrow x \cap \uparrow y$.

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(5) Let L be a complete antisymmetric non empty relational structure and X be a lower subset of L. If $\sup X \in X$, then $X = {\downarrow}\sup X$.

(6) Let L be a complete antisymmetric non empty relational structure and X be an upper subset of L. If $\inf X \in X$, then $X = {\uparrow}\inf X$.

(7) Let R be a non empty reflexive transitive relational structure and x, y be elements of R. Then $x \ll y$ if and only if ${\uparrow}y \subseteq {\Uparrow}x$.

(8) Let R be a non empty reflexive transitive relational structure and x, y be elements of R. Then $x \ll y$ if and only if ${\downarrow}x \subseteq {\Downarrow}y$.

(9) Let R be a complete reflexive antisymmetric non empty relational structure and x be an element of R. Then $\sup {\Downarrow}x \leqslant x$ and $x \leqslant \inf {\Uparrow}x$.

(10) For every lower-bounded antisymmetric non empty relational structure L holds ${\uparrow}(\bot_L) =$ the carrier of L.

(11) For every upper-bounded antisymmetric non empty relational structure L holds ${\downarrow}(\top_L) =$ the carrier of L.

(12) For every poset P with l.u.b.'s and for all elements x, y of P holds ${\Uparrow}x \sqcup {\Uparrow}y \subseteq {\uparrow}(x \sqcup y)$.

(13) For every poset P with g.l.b.'s and for all elements x, y of P holds ${\Downarrow}x \sqcap {\Downarrow}y \subseteq {\downarrow}(x \sqcap y)$.

(14) Let R be a non empty poset with l.u.b.'s and l be an element of R. Then l is co-prime if and only if for all elements x, y of R such that $l \leqslant x \sqcup y$ holds $l \leqslant x$ or $l \leqslant y$.

(15) For every complete non empty poset P and for every non empty subset V of P holds ${\downarrow}\inf V = \bigcap\{{\downarrow}u, u$ ranges over elements of P: $u \in V\}$.

(16) For every complete non empty poset P and for every non empty subset V of P holds ${\uparrow}\sup V = \bigcap\{{\uparrow}u, u$ ranges over elements of P: $u \in V\}$.

Let L be a sup-semilattice and let x be an element of L.

Note that compactbelow(x) is directed.

We now state four propositions:

(17) Let T be a non empty topological space, S be an irreducible subset of T, and V be an element of $\langle$the topology of T, $\subseteq\rangle$. If $V = -S$, then V is prime.

(18) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Then $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(19) Let T be a non empty topological space and V be an element of $\langle$the topology of T, $\subseteq\rangle$. Then V is prime if and only if for all elements X, Y of $\langle$the topology of T, $\subseteq\rangle$ such that $X \cap Y \subseteq V$ holds $X \subseteq V$ or $Y \subseteq V$.

(20) Let T be a non empty topological space and V be an element of $\langle$the topology of T, $\subseteq\rangle$. Then V is co-prime if and only if for all elements X, Y of $\langle$the topology of T, $\subseteq\rangle$ such that $V \subseteq X \cup Y$ holds $V \subseteq X$ or $V \subseteq Y$.

Let T be a non empty topological space. One can check that $\langle$the topology of T, $\subseteq\rangle$ is distributive.

The following propositions are true:

(21) Let T be a non empty topological space, L be a TopLattice, t be a point of T, l be a point of L, and X be a family of subsets of the carrier of L. Suppose the topological structure of T = the topological structure of L and $t = l$ and X is a basis of l. Then X is a basis of t.

(22) Let L be a TopLattice and x be an element of L. Suppose that for every subset X of L such that X is open holds X is upper. Then $\uparrow x$ is compact.

2. The Scott topology[2]

For simplicity, we use the following convention: L is a complete Scott TopLattice, x is an element of L, X, Y are subsets of L, V, W are elements of $\langle\sigma(L), \subseteq\rangle$, and V_1 is a subset of $\langle\sigma(L), \subseteq\rangle$.

Let L be a complete lattice. One can check that $\sigma(L)$ is non empty.

The following four propositions are true:

(23) $\sigma(L) =$ the topology of L.

(24) $X \in \sigma(L)$ iff X is open.

(25) For every filtered subset X of L such that $V_1 = \{-\downarrow x : x \in X\}$ holds V_1 is directed.

(26) If X is open and $x \in X$, then $\inf X \ll x$.

Let R be a non empty reflexive relational structure and let f be a map from $[\!: R, R :\!]$ into R. We say that f is jointly Scott-continuous if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let T be a non empty topological space. Suppose the topological structure of T = ConvergenceSpace(the Scott convergence of R). Then there exists a map f_1 from $[\!: T, T :\!]$ into T such that $f_1 = f$ and f_1 is continuous.

One can prove the following propositions:

(27) If $V = X$, then V is co-prime iff X is filtered and upper.

(28) If $V = X$ and there exists x such that $X = -\downarrow x$, then V is prime and $V \neq$ the carrier of L.

(29) If $V = X$ and $\sqcup_L$ is jointly Scott-continuous and V is prime and $V \neq$ the carrier of L, then there exists x such that $X = -\downarrow x$.

(30) If L is continuous, then $\sqcup_L$ is jointly Scott-continuous.

(31) If $\sqcup_L$ is jointly Scott-continuous, then L is sober.

[2] $\sigma(L) = \operatorname{sigma} L$, as defined in [34, p. 316, Def. 12] and $\sqcup_L = \text{sup\_op}(L)$, as defined in [21, p. 163, Def. 5].

(32) If L is continuous, then L is compact, locally-compact, sober, and Baire.

(33) If L is continuous and $X \in \sigma(L)$, then $X = \bigcup\{\uparrow x : x \in X\}$.

(34) If for every X such that $X \in \sigma(L)$ holds $X = \bigcup\{\uparrow x : x \in X\}$, then L is continuous.

(35) If L is continuous, then there exists a basis B of x such that for every X such that $X \in B$ holds X is open and filtered.

(36) If L is continuous, then $\langle \sigma(L), \subseteq \rangle$ is continuous.

(37) Suppose for every x there exists a basis B of x such that for every Y such that $Y \in B$ holds Y is open and filtered and $\langle \sigma(L), \subseteq \rangle$ is continuous. Then $x = \bigsqcup_L \{\inf X : x \in X \ \wedge \ X \in \sigma(L)\}$.

(38) If for every x holds $x = \bigsqcup_L \{\inf X : x \in X \ \wedge \ X \in \sigma(L)\}$, then L is continuous.

(39) The following statements are equivalent
  (i) for every x there exists a basis B of x such that for every Y such that $Y \in B$ holds Y is open and filtered,
  (ii) for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime.

(40) For every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime and $\langle \sigma(L), \subseteq \rangle$ is continuous if and only if $\langle \sigma(L), \subseteq \rangle$ is completely-distributive.

(41) $\langle \sigma(L), \subseteq \rangle$ is completely-distributive iff $\langle \sigma(L), \subseteq \rangle$ is continuous and $(\langle \sigma(L), \subseteq \rangle)^{\rm op}$ is continuous.

(42) If L is algebraic, then there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$.

(43) Given a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$. Then $\langle \sigma(L), \subseteq \rangle$ is algebraic and for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime.

(44) Suppose $\langle \sigma(L), \subseteq \rangle$ is algebraic and for every V there exists V_1 such that $V = \sup V_1$ and for every W such that $W \in V_1$ holds W is co-prime. Then there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$.

(45) If there exists a basis B of L such that $B = \{\uparrow x : x \in$ the carrier of CompactSublatt$(L)\}$, then L is algebraic.

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