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On the Categories Without Uniqueness of cod and dom . Some Properties of the Morphisms and the Functors

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The notation and terminology used here are introduced in the following papers: [9], [4], [10], [16], [2], [3], [1], [7], [8], [11], [15], [5], [12], [13], [6], and [14].

1. Preliminaries

In this paper C denotes a category and o_1, o_2, o_3 denote objects of C.

Let C be a non empty category structure with units and let o be an object of C. Observe that $\langle o, o \rangle$ is non empty.

The following propositions are true:

(1) Let v be a morphism from o_1 to o_2, u be a morphism from o_1 to o_3, and f be a morphism from o_2 to o_3. If $u = f \cdot v$ and $f^{-1} \cdot f = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_2 \rangle \neq \emptyset$, then $v = f^{-1} \cdot u$.

(2) Let v be a morphism from o_2 to o_3, u be a morphism from o_1 to o_3, and f be a morphism from o_1 to o_2. If $u = v \cdot f$ and $f \cdot f^{-1} = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$, then $v = u \cdot f^{-1}$.

(3) For every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is iso holds m^{-1} is iso.

(4) For every non empty category structure C with units and for every object o of C holds id_o is epi and mono.

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Let C be a non empty category structure with units and let o be an object of C. One can verify that id_o is epi mono retraction and coretraction.

Let C be a category and let o be an object of C. Note that id_o is iso.

We now state two propositions:

(5) Let f be a morphism from o_1 to o_2 and g, h be morphisms from o_2 to o_1. If $h \cdot f = \mathrm{id}_{(o_1)}$ and $f \cdot g = \mathrm{id}_{(o_2)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$, then $g = h$.

(6) Suppose that for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is coretraction. Let a, b be objects of C and g be a morphism from a to b. If $\langle a, b \rangle \neq \emptyset$ and $\langle b, a \rangle \neq \emptyset$, then g is iso.

2. Some properties of the initial and terminal objects

The following propositions are true:

(7) For all morphisms m, m' from o_1 to o_2 such that m is zero and m' is zero and there exists an object of C which is zero holds $m = m'$.

(8) Let C be a non empty category structure, O, A be objects of C, and M be a morphism from O to A. If O is terminal, then M is mono.

(9) Let C be a non empty category structure, O, A be objects of C, and M be a morphism from A to O. If O is initial, then M is epi.

(10) If o_2 is terminal and o_1, o_2 are iso , then o_1 is terminal.

(11) If o_1 is initial and o_1, o_2 are iso , then o_2 is initial.

(12) If o_1 is initial and o_2 is terminal and $\langle o_2, o_1 \rangle \neq \emptyset$, then o_2 is initial and o_1 is terminal.

3. The properties of the functors

One can prove the following propositions:

(13) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, and a be an object of A. Then $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

(14) Let C_1, C_2 be non empty category structures and F be a precontravariant functor structure from C_1 to C_2. Then F is full if and only if for all objects o_1, o_2 of C_1 holds Morph-Map$_F(o_2, o_1)$ is onto.

(15) Let C_1, C_2 be non empty category structures and F be a precontravariant functor structure from C_1 to C_2. Then F is faithful if and only if for all objects o_1, o_2 of C_1 holds Morph-Map$_F(o_2, o_1)$ is one-to-one.

(16) Let C_1, C_2 be non empty category structures, F be a precovariant functor structure from C_1 to C_2, o_1, o_2 be objects of C_1, and F_1 be a morphism from $F(o_1)$ to $F(o_2)$. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and F is full and feasible. Then there exists a morphism m from o_1 to o_2 such that $F_1 = F(m)$.

(17) Let C_1, C_2 be non empty category structures, F be a precontravariant functor structure from C_1 to C_2, o_1, o_2 be objects of C_1, and F_1 be a morphism from $F(o_2)$ to $F(o_1)$. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and F is full and feasible. Then there exists a morphism m from o_1 to o_2 such that $F_1 = F(m)$.

(18) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is retraction, then $F(a)$ is retraction.

(19) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is coretraction, then $F(a)$ is coretraction.

(20) Let A, B be categories, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is iso, then $F(a)$ is iso.

(21) Let A, B be categories, F be a covariant functor from A to B, and o_1, o_2 be objects of A. If o_1, o_2 are iso , then $F(o_1)$, $F(o_2)$ are iso .

(22) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is retraction, then $F(a)$ is coretraction.

(23) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is coretraction, then $F(a)$ is retraction.

(24) Let A, B be categories, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and a is iso, then $F(a)$ is iso.

(25) Let A, B be categories, F be a contravariant functor from A to B, and o_1, o_2 be objects of A. If o_1, o_2 are iso , then $F(o_2)$, $F(o_1)$ are iso .

(26) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is retraction. Then a is retraction.

(27) Let A, B be transitive non empty category structures with units, F

be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is coretraction. Then a is coretraction.

(28) Let A, B be categories, F be a covariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is iso. Then a is iso.

(29) Let A, B be categories, F be a covariant functor from A to B, and o_1, o_2 be objects of A. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(o_1)$, $F(o_2)$ are iso . Then o_1, o_2 are iso .

(30) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is retraction. Then a is coretraction.

(31) Let A, B be transitive non empty category structures with units, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is coretraction. Then a is retraction.

(32) Let A, B be categories, F be a contravariant functor from A to B, o_1, o_2 be objects of A, and a be a morphism from o_1 to o_2. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(a)$ is iso. Then a is iso.

(33) Let A, B be categories, F be a contravariant functor from A to B, and o_1, o_2 be objects of A. Suppose F is full and faithful and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $F(o_2)$, $F(o_1)$ are iso . Then o_1, o_2 are iso .

4. The subcategories of the morphisms

We now state two propositions:

(34) Let C be a category structure and D be a substructure of C. Suppose the carrier of C = the carrier of D and the arrows of C = the arrows of D. Then D is full.

(35) Let C be a non empty category structure with units and D be a substructure of C. Suppose the carrier of C = the carrier of D and the arrows of C = the arrows of D. Then D is full and id-inheriting.

Let C be a category. Observe that there exists a subcategory of C which is full, non empty, and strict.

Next we state several propositions:

(36) For every non empty subcategory B of C holds every non empty subcategory of B is a non empty subcategory of C.

(37) Let C be a non empty transitive category structure, D be a non empty transitive substructure of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$. Then

(i) if m is mono, then n is mono, and

(ii) if m is epi, then n is epi.

(38) Let D be a non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, m_1 be a morphism from o_2 to o_1, n be a morphism from p_1 to p_2, and n_1 be a morphism from p_2 to p_1 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $m_1 = n_1$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) m is left inverse of m_1 iff n is left inverse of n_1, and

(ii) m is right inverse of m_1 iff n is right inverse of n_1.

(39) Let D be a full non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) if m is retraction, then n is retraction,

(ii) if m is coretraction, then n is coretraction, and

(iii) if m is iso, then n is iso.

(40) Let D be a non empty subcategory of C, o_1, o_2 be objects of C, p_1, p_2 be objects of D, m be a morphism from o_1 to o_2, and n be a morphism from p_1 to p_2 such that $p_1 = o_1$ and $p_2 = o_2$ and $m = n$ and $\langle p_1, p_2 \rangle \neq \emptyset$ and $\langle p_2, p_1 \rangle \neq \emptyset$. Then

(i) if n is retraction, then m is retraction,

(ii) if n is coretraction, then m is coretraction, and

(iii) if n is iso, then m is iso.

Let C be a category. The functor AllMono C yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of AllMono C = the carrier of C,

(ii) the arrows of AllMono $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllMono C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and m is mono.

Let C be a category. Note that AllMono C is id-inheriting.

Let C be a category. The functor AllEpi C yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of AllEpi C = the carrier of C,

(ii) the arrows of AllEpi $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllEpi C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and m is epi.

Let C be a category. Observe that AllEpi C is id-inheriting.

Let C be a category. The functor AllRetr C yielding a strict non empty transitive substructure of C is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of AllRetr $C =$ the carrier of C,

(ii) the arrows of AllRetr $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllRetr C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is retraction.

Let C be a category. One can check that AllRetr C is id-inheriting.

Let C be a category. The functor AllCoretr C yielding a strict non empty transitive substructure of C is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of AllCoretr $C =$ the carrier of C,

(ii) the arrows of AllCoretr $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllCoretr C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is coretraction.

Let C be a category. One can verify that AllCoretr C is id-inheriting.

Let C be a category. The functor AllIso C yields a strict non empty transitive substructure of C and is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of AllIso $C =$ the carrier of C,

(ii) the arrows of AllIso $C \mathrel{\dot{\subseteq}}$ the arrows of C, and

(iii) for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds $m \in$ (the arrows of AllIso C)(o_1, o_2) iff $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and m is iso.

Let C be a category. Note that AllIso C is id-inheriting.

Next we state a number of propositions:

(41) AllIso C is a non empty subcategory of AllRetr C.

(42) AllIso C is a non empty subcategory of AllCoretr C.

(43) AllCoretr C is a non empty subcategory of AllMono C.

(44) AllRetr C is a non empty subcategory of AllEpi C.

(45) If for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is mono, then the category structure of $C =$ AllMono C.

(46) If for all objects o_1, o_2 of C holds every morphism from o_1 to o_2 is epi, then the category structure of $C =$ AllEpi C.

(47) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is retraction and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of $C =$ AllRetr C.

(48) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is coretraction and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of $C =$ AllCoretr C.

(49) Suppose that for all objects o_1, o_2 of C and for every morphism m from o_1 to o_2 holds m is iso and $\langle o_2, o_1 \rangle \neq \emptyset$. Then the category structure of $C = \mathrm{AllIso}\, C$.

(50) For all objects o_1, o_2 of $\mathrm{AllMono}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is mono.

(51) For all objects o_1, o_2 of $\mathrm{AllEpi}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is epi.

(52) For all objects o_1, o_2 of $\mathrm{AllIso}\, C$ and for every morphism m from o_1 to o_2 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds m is iso and $m^{-1} \in \langle o_2, o_1 \rangle$.

(53) $\mathrm{AllMono}\, \mathrm{AllMono}\, C = \mathrm{AllMono}\, C$.

(54) $\mathrm{AllEpi}\, \mathrm{AllEpi}\, C = \mathrm{AllEpi}\, C$.

(55) $\mathrm{AllIso}\, \mathrm{AllIso}\, C = \mathrm{AllIso}\, C$.

(56) $\mathrm{AllIso}\, \mathrm{AllMono}\, C = \mathrm{AllIso}\, C$.

(57) $\mathrm{AllIso}\, \mathrm{AllEpi}\, C = \mathrm{AllIso}\, C$.

(58) $\mathrm{AllIso}\, \mathrm{AllRetr}\, C = \mathrm{AllIso}\, C$.

(59) $\mathrm{AllIso}\, \mathrm{AllCoretr}\, C = \mathrm{AllIso}\, C$.

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