FORMALIZED MATHEMATICS
Volume 6, Number 4, 1997
University of Białystok

The Ordering of Points on a Curve. Part I

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Summary. Some auxiliary theorems needed to formalize the proof of the Jordan Curve Theorem according to [25] are proved.

MML Identifier: JORDAN5B.

The articles [26], [29], [13], [1], [22], [24], [31], [2], [4], [5], [11], [28], [20], [12], [16], [23], [9], [8], [27], [10], [30], [15], [17], [18], [14], [19], [21], [6], [7], and [3] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For every natural number i_1 such that $1 \leqslant i_1$ holds $i_1 -' 1 < i_1$.

(2) For all natural numbers i, k such that $i + 1 \leqslant k$ holds $1 \leqslant k -' i$.

(3) For all natural numbers i, k such that $1 \leqslant i$ and $1 \leqslant k$ holds $k -' i + 1 \leqslant k$.

(4) For every real number r such that $r \in$ the carrier of $\mathbb{I}$ holds $1 - r \in$ the carrier of $\mathbb{I}$.

(5) For all points p, q, p_1 of $\mathcal{E}^2_{\rm T}$ such that $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p_1 \in \mathcal{L}(p, q)$ holds if $(p_1)_{\mathbf{2}} = p_{\mathbf{2}}$, then $(p_1)_{\mathbf{1}} = p_{\mathbf{1}}$.

(6) For all points p, q, p_1 of $\mathcal{E}^2_{\rm T}$ such that $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_1 \in \mathcal{L}(p, q)$ holds if $(p_1)_{\mathbf{1}} = p_{\mathbf{1}}$, then $(p_1)_{\mathbf{2}} = p_{\mathbf{2}}$.

[1]This paper was written while the author visited the Shinshu University in the winter of 1997.

ISSN 1426–2630

(7) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}){\restriction}P$, and i be a natural number. Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and F is a homeomorphism and $F(0) = \pi_1 f$ and $F(1) = \pi_{\operatorname{len} f} f$. Then there exist real numbers p_1, p_2 such that $p_1 < p_2$ and $0 \leqslant p_1$ and $p_1 \leqslant 1$ and $0 \leqslant p_2$ and $p_2 \leqslant 1$ and $\mathcal{L}(f, i) = F^{\circ}[p_1, p_2]$ and $F(p_1) = \pi_i f$ and $F(p_2) = \pi_{i+1} f$.

(8) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q, R be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, F be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}){\restriction}Q$, i be a natural number, and P be a non empty subset of $\mathbb{I}$. Suppose that

(i) f is a special sequence,

(ii) F is a homeomorphism,

(iii) $F(0) = \pi_1 f$,

(iv) $F(1) = \pi_{\operatorname{len} f} f$,

(v) $1 \leqslant i$,

(vi) $i+1 \leqslant \operatorname{len} f$,

(vii) $F^{\circ}P = \mathcal{L}(f, i)$,

(viii) $Q = \widetilde{\mathcal{L}}(f)$, and

(ix) $R = \mathcal{L}(f, i)$.

Then there exists a map G from $\mathbb{I}{\restriction}P$ into $(\mathcal{E}^2_{\mathrm{T}}){\restriction}R$ such that $G = F{\restriction}P$ and G is a homeomorphism.

2. Some properties of real intervals

One can prove the following propositions:

(9) For all points p_1, p_2, p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $\mathrm{LE}(p, p, p_1, p_2)$.

(10) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$ holds $\mathrm{LE}(p_1, p, p_1, p_2)$.

(11) For all points p, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $\mathrm{LE}(p, p_2, p_1, p_2)$.

(12) For all points p_1, p_2, q_1, q_2, q_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ and $\mathrm{LE}(q_1, q_2, p_1, p_2)$ and $\mathrm{LE}(q_2, q_3, p_1, p_2)$ holds $\mathrm{LE}(q_1, q_3, p_1, p_2)$.

(13) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ holds $\mathcal{L}(p, q) = \{p_1; p_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathrm{LE}(p, p_1, p, q) \ \wedge\ \mathrm{LE}(p_1, q, p, q)\}$.

(14) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then P is an arc from p_2 to p_1.

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose f is a special sequence and

$1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $P = \mathcal{L}(f, i)$. Then P is an arc from $\pi_i f$ to $\pi_{i+1} f$.

3. Cutting off sequences

One can prove the following propositions:

(16) Let g_1 be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i be a natural number. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} g_1$ and g_1 is a special sequence. If $\pi_1 g_1 \in \widetilde{\mathcal{L}}(\operatorname{mid}(g_1, i, \operatorname{len} g_1))$, then $i = 1$.

(17) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p = f(\operatorname{len} f)$, then $\rfloor p, f = \langle p, p \rangle$.

(18) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k \leqslant \operatorname{len} f$, then $\operatorname{mid}(f, k, k) = \langle \pi_k f \rangle$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p = f(1)$, then $\lfloor f, p = \langle p \rangle$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $\widetilde{\mathcal{L}}(\lfloor f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ and f is a special sequence, then $\operatorname{Index}(p, \rfloor p, f) = 1$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

(23) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(1)$, then $p \in \widetilde{\mathcal{L}}(\lfloor f, p)$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ and f is a special sequence, then $\rfloor \lfloor p, f, p = \langle p \rangle$.

(25) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p = f(\operatorname{len} f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\rfloor q, f)$.

(26) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $p \in \widetilde{\mathcal{L}}(\rfloor q, f)$ or $q \in \widetilde{\mathcal{L}}(\rfloor p, f)$.

(27) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\operatorname{len} f)$ or $q \neq f(\operatorname{len} f)$ and f is a special sequence. Then $\widetilde{\mathcal{L}}(\rfloor \lfloor p, f, q) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leqslant i$ and $j \leqslant \operatorname{len}$ the Go-board of f and $i < j$. Then $\mathcal{L}((\text{the Go-board of } f)_{1, \text{width the Go-board of } f}, (\text{the Go-board of } f)_{i, \text{width the Go-board of } f}) \cap \mathcal{L}((\text{the Go-board of } f)_{j, \text{width the Go-board of } f}, (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f}) = \emptyset$.

(29) Let f be a non constant standard special circular sequence and i, j be natural numbers. Suppose $1 \leqslant i$ and $j \leqslant$ width the Go-board of f and $i < j$. Then $\mathcal{L}(($the Go-board of $f)_{\text{len the Go-board of } f,\, 1}, ($the Go-board of $f)_{\text{len the Go-board of } f,\, i}) \cap \mathcal{L}(($the Go-board of $f)_{\text{len the Go-board of } f,\, j}, ($the Go-board of $f)_{\text{len the Go-board of } f,\, \text{width the Go-board of } f}) = \emptyset$.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence, then $\downharpoonleft \pi_1 f, f = f$.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence, then $\downharpoonright f, \pi_{\text{len} f} f = f$.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $p \neq f(\text{len} f)$, then $p \in \mathcal{L}(\pi_{\text{Index}(p,f)} f, \pi_{\text{Index}(p,f)+1} f)$.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. If f is a special sequence, then if $\pi_1 f \in \mathcal{L}(f, i)$, then $i = 1$.

(34) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. Suppose $1 \leqslant j$ and $j \leqslant$ width the Go-board of f and $P = \mathcal{L}(($the Go-board of $f)_{1,j}, ($the Go-board of $f)_{\text{len the Go-board of } f,\, j})$. Then P is a special polygonal arc joining (the Go-board of $f)_{1,j}$ and (the Go-board of $f)_{\text{len the Go-board of } f,\, j}$.

(35) Let f be a non constant standard special circular sequence, j be a natural number, and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. Suppose $1 \leqslant j$ and $j \leqslant$ len the Go-board of f and $P = \mathcal{L}(($the Go-board of $f)_{j,1}, ($the Go-board of $f)_{j,\text{width the Go-board of } f})$. Then P is a special polygonal arc joining (the Go-board of $f)_{j,1}$ and (the Go-board of $f)_{j,\text{width the Go-board of } f}$.

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Received September 10, 1997