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The Jónson's Theorem

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The papers [30], [16], [34], [36], [35], [13], [14], [6], [33], [29], [21], [26], [2], [18], [23], [3], [4], [1], [31], [28], [22], [15], [19], [24], [27], [32], [25], [20], [10], [12], [5], [17], [37], [7], [11], [8], [9], and [38] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *RecChoice* deals with a set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the following condition is satisfied:

- For every natural number n and for every set x there exists a set y such that $\mathcal{P}[n, x, y]$.

One can prove the following propositions:

(1) For every function f and for every function yielding function F such that $f = \bigcup \operatorname{rng} F$ holds $\operatorname{dom} f = \bigcup \operatorname{rng}(\operatorname{dom}_\kappa F(\kappa))$.

(2) For all non empty sets A, B holds $[\!: \bigcup A, \bigcup B :\!] = \bigcup \{[\!: a, b :\!]; a$ ranges over elements of A, b ranges over elements of B: $a \in A \ \wedge \ b \in B\}$.

(3) For every non empty set A such that A is $\subseteq$-linear holds $[\!: \bigcup A, \bigcup A :\!] = \bigcup \{[\!: a, a :\!]; a$ ranges over elements of A: $a \in A\}$.

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2. An equivalence lattice of a set

In the sequel X is a non empty set.

Let A be a non empty set. The functor EqRelPoset(A) yielding a poset is defined as follows:

(Def. 1) EqRelPoset(A) = Poset(EqRelLatt(A)).

Let A be a non empty set. One can check that EqRelPoset(A) is non empty and has g.l.b.'s and l.u.b.'s.

One can prove the following propositions:

(4) Let A be a non empty set and x be a set. Then $x \in$ the carrier of EqRelPoset(A) if and only if x is an equivalence relation of A.

(5) For every non empty set A and for all elements x, y of the carrier of EqRelLatt(A) holds $x \sqsubseteq y$ iff $x \subseteq y$.

(6) For every non empty set A and for all elements a, b of EqRelPoset(A) holds $a \leqslant b$ iff $a \subseteq b$.

(7) For every lattice L and for all elements a, b of Poset(L) holds $a \sqcap b = {}^{\cdot}a \sqcap {}^{\cdot}b$.

(8) For every non empty set A and for all elements a, b of EqRelPoset(A) holds $a \sqcap b = a \cap b$.

(9) For every lattice L and for all elements a, b of Poset(L) holds $a \sqcup b = {}^{\cdot}a \sqcup {}^{\cdot}b$.

(10) Let A be a non empty set, a, b be elements of EqRelPoset(A), and E_1, E_2 be equivalence relations of A. If $a = E_1$ and $b = E_2$, then $a \sqcup b = E_1 \sqcup E_2$.

(11) Let L be a lattice, X be a set, and b be an element of L. Then $b \leqslant X$ if and only if $b \leqslant X \cap$ the carrier of L.

Let L be a non empty relational structure. Let us observe that L is complete if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let X be a subset of L. Then there exists an element a of L such that $a \leqslant X$ and for every element b of L such that $b \leqslant X$ holds $b \leqslant a$.

Let A be a non empty set. Note that EqRelPoset(A) is complete.

3. A type of a sublattice of equivalence lattice of a set

Let L_1, L_2 be lattices. One can check that there exists a map from L_1 into L_2 which is meet-preserving and join-preserving.

Let L_1, L_2 be lattices. A homomorphism from L_1 to L_2 is a meet-preserving join-preserving map from L_1 into L_2.

Let L be a lattice. One can check that there exists a relational substructure of L which is meet-inheriting, join-inheriting, and strict.

Let L_1, L_2 be lattices and let f be a homomorphism from L_1 to L_2. Then $\mathrm{Im}\, f$ is a strict full sublattice of L_2.

We follow the rules: e, e_1, e_2 denote equivalence relations of X and x, y denote sets.

Let us consider X, let f be a non empty finite sequence of elements of X, let us consider x, y, and let R be a binary relation. We say that x and y are joint by f and R if and only if:

(Def. 3) $f(1) = x$ and $f(\mathrm{len}\, f) = y$ and for every natural number i such that $1 \leqslant i$ and $i < \mathrm{len}\, f$ holds $\langle f(i), f(i+1)\rangle \in R$.

One can prove the following propositions:

(12) Let x be a set, o be a natural number, R be a binary relation, and f be a non empty finite sequence of elements of X. If R is reflexive in X and $f = o \mapsto x$, then x and x are joint by f and R.

(13) Let x, y, z be sets, R be a binary relation, and f, g be non empty finite sequences of elements of X. Suppose R is reflexive in X and x and y are joint by f and R and y and z are joint by g and R. Then there exists a non empty finite sequence h of elements of X such that $h = f \frown g$ and x and z are joint by h and R.

(14) Let x, y be sets, R be a binary relation, and n, m be natural numbers. Suppose that

(i) $n \leqslant m$,

(ii) R is reflexive in X, and

(iii) there exists a non empty finite sequence f of elements of X such that $\mathrm{len}\, f = n$ and x and y are joint by f and R.

Then there exists a non empty finite sequence h of elements of X such that $\mathrm{len}\, h = m$ and x and y are joint by h and R.

Let us consider X and let Y be a sublattice of $\mathrm{EqRelPoset}(X)$. Let us assume that there exists e such that $e \in$ the carrier of Y $e \neq \mathrm{id}_X$. And let us assume that there exists a natural number o such that for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\mathrm{len}\, F = o$ and x and y are joint by F and $e_1 \cup e_2$. The type of Y is a natural number and is defined by the conditions (Def. 4).

(Def. 4)(i) For all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\mathrm{len}\, F =$ (the type of Y) $+ 2$ and x and y are joint by F and $e_1 \cup e_2$, and

(ii) there exist e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ and it is not true that there exists a non empty finite sequence F of elements of X such that $\mathrm{len}\, F =$ (the type of Y) $+ 1$ and x and y are joint by F and $e_1 \cup e_2$.

One can prove the following proposition

(15) Let Y be a sublattice of EqRelPoset(X) and n be a natural number. Suppose that

(i) there exists e such that $e \in$ the carrier of Y and $e \neq \mathrm{id}_X$, and

(ii) for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\mathrm{len}\, F = n + 2$ and x and y are joint by F and $e_1 \cup e_2$.

Then the type of $Y \leqslant n$.

4. A meet-representation of a lattice

In the sequel A is a non empty set and L is a lower-bounded lattice.

Let us consider A, L.

(Def. 5) A function from $[\!:A, A:\!]$ into the carrier of L is said to be a bifunction from A into L.

Let us consider A, L, let f be a bifunction from A into L, and let x, y be elements of A. Then $f(x, y)$ is an element of L.

Let us consider A, L and let f be a bifunction from A into L. We say that f is symmetric if and only if:

(Def. 6) For all elements x, y of A holds $f(x, y) = f(y, x)$.

We say that f is zeroed if and only if:

(Def. 7) For every element x of A holds $f(x, x) = \bot_L$.

We say that f satisfies triangle inequality if and only if:

(Def. 8) For all elements x, y, z of A holds $f(x, y) \sqcup f(y, z) \geqslant f(x, z)$.

Let us consider A, L. Observe that there exists a bifunction from A into L which is symmetric and zeroed and satisfies triangle inequality.

Let us consider A, L. A distance function of A, L is a symmetric zeroed bifunction from A into L satisfying triangle inequality.

Let us consider A, L and let d be a distance function of A, L. The functor $\alpha(d)$ yielding a map from L into EqRelPoset(A) is defined by the condition (Def. 9).

(Def. 9) Let e be an element of L. Then there exists an equivalence relation E of A such that $E = (\alpha(d))(e)$ and for all elements x, y of A holds $\langle x, y\rangle \in E$ iff $d(x, y) \leqslant e$.

The following two propositions are true:

(16) For every distance function d of A, L holds $\alpha(d)$ is meet-preserving.

(17) For every distance function d of A, L such that d is onto holds $\alpha(d)$ is one-to-one.

5. Jónson's theorem

Let A be a set. The functor A^* is defined as follows:

(Def. 10) $A^* = A \cup \{\{A\}, \{\{A\}\}, \{\{\{A\}\}\}\}$.

Let A be a set. One can verify that A^* is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be an element of $[\![A, A, \text{the carrier of } L, \text{ the carrier of } L]\!]$. The functor d_q^* yields a bifunction from A^* into L and is defined by the conditions (Def. 11).

(Def. 11)(i) For all elements u, v of A holds $d_q^*(u, v) = d(u, v)$,

(ii) $d_q^*(\{A\}, \{A\}) = \bot_L$,

(iii) $d_q^*(\{\{A\}\}, \{\{A\}\}) = \bot_L$,

(iv) $d_q^*(\{\{\{A\}\}\}, \{\{\{A\}\}\}) = \bot_L$,

(v) $d_q^*(\{\{A\}\}, \{\{\{A\}\}\}) = q_{\mathbf{3}}$,

(vi) $d_q^*(\{\{\{A\}\}\}, \{\{A\}\}) = q_{\mathbf{3}}$,

(vii) $d_q^*(\{A\}, \{\{A\}\}) = q_{\mathbf{4}}$,

(viii) $d_q^*(\{\{A\}\}, \{A\}) = q_{\mathbf{4}}$,

(ix) $d_q^*(\{A\}, \{\{\{A\}\}\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$,

(x) $d_q^*(\{\{\{A\}\}\}, \{A\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$, and

(xi) for every element u of A holds $d_q^*(u, \{A\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(\{A\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(u, \{\{A\}\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{A\}\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(u, \{\{\{A\}\}\}) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{\{A\}\}\}, u) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$.

Next we state several propositions:

(18) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be an element of $[\![A, A, \text{the carrier of } L, \text{ the carrier of } L]\!]$. Then d_q^* is zeroed.

(19) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be an element of $[\![A, A, \text{the carrier of } L, \text{ the carrier of } L]\!]$. Then d_q^* is symmetric.

(20) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be an element of $[\![A, A, \text{the carrier of } L, \text{ the carrier of } L]\!]$. If $d(q_{\mathbf{1}}, q_{\mathbf{2}}) \leqslant q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$, then d_q^* satisfies triangle inequality.

(21) For every set A holds $A \subseteq A^*$.

(22) Let d be a bifunction from A into L and q be an element of $[\![A, A, \text{the carrier of } L, \text{ the carrier of } L]\!]$. Then $d \subseteq d_q^*$.

Let us consider A, L and let d be a bifunction from A into L. The functor DistEsti(d) yields a cardinal number and is defined as follows:

(Def. 12) $\text{DistEsti}(d) \approx \{\langle x, y, a, b \rangle; x$ ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x, y) \leqslant a \sqcup b\}$.

We now state the proposition

(23) For every distance function d of A, L holds $\mathrm{DistEsti}(d) \neq \emptyset$.

In the sequel T denotes a transfinite sequence and O, O_1, O_2 denote ordinal numbers.

Let us consider A and let us consider O. The functor $\mathrm{ConsecutiveSet}(A, O)$ is defined by the condition (Def. 13).

(Def. 13) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveSet}(A, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = A$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = z^*$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

We now state three propositions:

(24) $\mathrm{ConsecutiveSet}(A, \emptyset) = A$.

(25) $\mathrm{ConsecutiveSet}(A, \mathrm{succ}\, O) = (\mathrm{ConsecutiveSet}(A, O))^*$.

(26) Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveSet}(A, O_1)$. Then $\mathrm{ConsecutiveSet}(A, O) = \bigcup \mathrm{rng}\, T$.

Let us consider A and let us consider O. Note that $\mathrm{ConsecutiveSet}(A, O)$ is non empty.

One can prove the following proposition

(27) $A \subseteq \mathrm{ConsecutiveSet}(A, O)$.

Let us consider A, L and let d be a bifunction from A into L. A transfinite sequence of elements of $[\!:A, A, \text{the carrier of } L, \text{the carrier of } L:\!]$ is said to be a sequence of quadruples of d if it satisfies the conditions (Def. 14).

(Def. 14)(i) dom it is a cardinal number,

(ii) it is one-to-one, and

(iii) $\mathrm{rng}\, \mathrm{it} = \{\langle x, y, a, b\rangle;$ x ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x, y) \leqslant a \sqcup b\}$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Let us assume that $O \in \mathrm{dom}\, q$. The functor $\mathrm{Quadr}(q, O)$ yielding an element of $[\!:\mathrm{ConsecutiveSet}(A, O), \mathrm{ConsecutiveSet}(A, O), \text{the carrier of } L, \text{the carrier of } L:\!]$ is defined as follows:

(Def. 15) $\mathrm{Quadr}(q, O) = q(O)$.

One can prove the following proposition

(28) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $O \in \mathrm{DistEsti}(d)$ if and only if $O \in \mathrm{dom}\, q$.

Let us consider A, L and let z be a set. Let us assume that z is a bifunction from A into L. The functor $\mathrm{BiFun}(z, A, L)$ yields a bifunction from A into L and is defined as follows:

(Def. 16) $\mathrm{BiFun}(z, A, L) = z$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. The functor $\mathrm{ConsecutiveDelta}(q, O)$ is defined by the condition (Def. 17).

(Def. 17) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveDelta}(q, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = d$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = (\mathrm{BiFun}(z, \mathrm{ConsecutiveSet}(A, C), L))^*_{\mathrm{Quadr}(q,C)}$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

Next we state four propositions:

(29) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q, \emptyset) = d$.

(30) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q, \mathrm{succ}\, O) = (\mathrm{BiFun}(\mathrm{ConsecutiveDelta}(q, O), \mathrm{ConsecutiveSet}(A, O), L))^*_{\mathrm{Quadr}(q,O)}$.

(31) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveDelta}(q, O_1)$. Then $\mathrm{ConsecutiveDelta}(q, O) = \bigcup \mathrm{rng}\, T$.

(32) If $O_1 \subseteq O_2$, then $\mathrm{ConsecutiveSet}(A, O_1) \subseteq \mathrm{ConsecutiveSet}(A, O_2)$.

Let O be a non empty ordinal number. Note that every element of O is ordinal-like.

Next we state the proposition

(33) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

Next we state several propositions:

(34) For every bifunction d from A into L and for every sequence q of quadruples of d holds $d \subseteq \text{ConsecutiveDelta}(q, O)$.

(35) For every bifunction d from A into L and for every sequence q of quadruples of d such that $O_1 \subseteq O_2$ holds $\text{ConsecutiveDelta}(q, O_1) \subseteq \text{ConsecutiveDelta}(q, O_2)$.

(36) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be a sequence of quadruples of d. Then $\text{ConsecutiveDelta}(q, O)$ is zeroed.

(37) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be a sequence of quadruples of d. Then $\text{ConsecutiveDelta}(q, O)$ is symmetric.

(38) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be a sequence of quadruples of d. If $O \subseteq \text{DistEsti}(d)$, then $\text{ConsecutiveDelta}(q, O)$ satisfies triangle inequality.

(39) Let d be a distance function of A, L and q be a sequence of quadruples of d. If $O \subseteq \text{DistEsti}(d)$, then $\text{ConsecutiveDelta}(q, O)$ is a distance function of $\text{ConsecutiveSet}(A, O)$, L.

Let us consider A, L and let d be a bifunction from A into L. The functor $\text{NextSet}(d)$ is defined as follows:

(Def. 18) $\text{NextSet}(d) = \text{ConsecutiveSet}(A, \text{DistEsti}(d))$.

Let us consider A, L and let d be a bifunction from A into L. One can check that $\text{NextSet}(d)$ is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be a sequence of quadruples of d. The functor $\text{NextDelta}(q)$ is defined as follows:

(Def. 19) $\text{NextDelta}(q) = \text{ConsecutiveDelta}(q, \text{DistEsti}(d))$.

Let us consider A, L, let d be a distance function of A, L, and let q be a sequence of quadruples of d. Then $\text{NextDelta}(q)$ is a distance function of $\text{NextSet}(d)$, L.

Let us consider A, L, let d be a distance function of A, L, let A_1 be a non empty set, and let d_1 be a distance function of A_1, L. We say that (A_1, d_1) is extension of (A, d) if and only if:

(Def. 20) There exists a sequence q of quadruples of d such that $A_1 = \text{NextSet}(d)$ and $d_1 = \text{NextDelta}(q)$.

The following proposition is true

(40) Let d be a distance function of A, L, A_1 be a non empty set, and d_1 be a distance function of A_1, L. Suppose (A_1, d_1) is extension of (A, d). Let x, y be elements of A and a, b be elements of L. Suppose $d(x, y) \leqslant a \sqcup b$. Then there exist elements z_1, z_2, z_3 of A_1 such that $d_1(x, z_1) = a$ and $d_1(z_2, z_3) = a$ and $d_1(z_1, z_2) = b$ and $d_1(z_3, y) = b$.

Let us consider A, L and let d be a distance function of A, L. A function is called an extension sequence of (A, d) if it satisfies the conditions (Def. 21).

(Def. 21)(i) $\text{dom}\,\text{it} = \mathbb{N}$,

(ii) $\mathrm{it}(0) = \langle A, d \rangle$, and

(iii) for every natural number n there exists a non empty set A' and there exists a distance function d' of A', L and there exists a non empty set A_1 and there exists a distance function d_1 of A_1, L such that (A_1, d_1) is extension of (A', d') and $\mathrm{it}(n) = \langle A', d' \rangle$ and $\mathrm{it}(n+1) = \langle A_1, d_1 \rangle$.

Next we state two propositions:

(41) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{1}} \subseteq S(l)_{\mathbf{1}}$.

(42) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{2}} \subseteq S(l)_{\mathbf{2}}$.

Let us consider L. The functor $\delta_0(L)$ yields a distance function of the carrier of L, L and is defined by:

(Def. 22) For all elements x, y of the carrier of L holds if $x \neq y$, then $(\delta_0(L))(x, y) = x \sqcup y$ and if $x = y$, then $(\delta_0(L))(x, y) = \bot_L$.

We now state two propositions:

(43) $\delta_0(L)$ is onto.

(44) There exists a non empty set A and there exists a homomorphism f from L to $\mathrm{EqRelPoset}(A)$ such that f is one-to-one and the type of $\mathrm{Im}\, f \leqslant 3$.

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