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On the Order on a Special Polygon

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Summary. The goal of the article is to determine the order of the special points defined in [10] on a special polygon. We restrict ourselves to the clockwise oriented finite sequences (the concept defined in this article) that start in N-min C (C being a compact non empty subset of the plane).

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The papers [28], [33], [27], [7], [15], [29], [34], [1], [5], [6], [3], [32], [8], [30], [16], [17], [2], [25], [4], [19], [18], [26], [11], [12], [13], [14], [21], [20], [22], [9], [24], [23], [10], and [31] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all sets A, B, C, p such that $A \cap B \subseteq \{p\}$ and $p \in C$ and C misses B holds $A \cup C$ misses B.

(2) For all sets A, B, C, p such that $A \cap C = \{p\}$ and $p \in B$ and $B \subseteq C$ holds $A \cap B = \{p\}$.

(3) For all sets A, B such that for every set y such that $y \in B$ holds A misses y holds A misses $\bigcup B$.

(4) For all sets A, B such that for all sets x, y such that $x \in A$ and $y \in B$ holds x misses y holds $\bigcup A$ misses $\bigcup B$.

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2. On the finite sequences

We adopt the following convention: i, j, k, m, n denote natural numbers, D denotes a non empty set, and f denotes a finite sequence of elements of D.

The following propositions are true:

(5) For all i, j, k such that $i \leqslant j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $(k + i) -' 1 \in \operatorname{dom} f$.

(6) For all i, j, k such that $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $i -' k + 1 \in \operatorname{dom} f$.

(7) For all i, j, k such that $i \leqslant j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $\pi_k \operatorname{mid}(f, i, j) = \pi_{(k+i)-'1} f$.

(8) For all i, j, k such that $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f, i, j)$ holds $\pi_k \operatorname{mid}(f, i, j) = \pi_{i-'k+1} f$.

(9) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{len} \operatorname{mid}(f, i, j) \geqslant 1$.

(10) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{len} \operatorname{mid}(f, i, j) = 1$, then $i = j$.

(11) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f, i, j)$ is non empty.

(12) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\pi_1 \operatorname{mid}(f, i, j) = \pi_i f$.

(13) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\pi_{\operatorname{len} \operatorname{mid}(f,i,j)} \operatorname{mid}(f, i, j) = \pi_j f$.

3. Compact subsets of the plane

In the sequel X denotes a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following four propositions:

(14) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{N-bound}\, X$ holds $p \in \text{N-most}\, X$.

(15) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{S-bound}\, X$ holds $p \in \text{S-most}\, X$.

(16) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{W-bound}\, X$ holds $p \in \text{W-most}\, X$.

(17) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{E-bound}\, X$ holds $p \in \text{E-most}\, X$.

4. Finite sequences on the plane

We now state several propositions:

(18) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leqslant i$ and $i \leqslant j$ and $j \leqslant \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j)) = \bigcup\{\mathcal{L}(f,k) : i \leqslant k \ \wedge \ k < j\}$.

(19) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{dom} \mathbf{X}\text{-coordinate}(f) = \operatorname{dom} f$.

(20) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{dom} \mathbf{Y}\text{-coordinate}(f) = \operatorname{dom} f$.

(21) For all points a, b, c of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \leqslant b_{\mathbf{1}}$ and $c_{\mathbf{1}} \leqslant b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(22) For all points a, b, c of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \leqslant b_{\mathbf{2}}$ and $c_{\mathbf{2}} \leqslant b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

(23) For all points a, b, c of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \geqslant b_{\mathbf{1}}$ and $c_{\mathbf{1}} \geqslant b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(24) For all points a, b, c of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \geqslant b_{\mathbf{2}}$ and $c_{\mathbf{2}} \geqslant b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

5. The area of a sequence

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that g is in the area of f if and only if:

(Def. 1) For every n such that $n \in \operatorname{dom} g$ holds $\text{W-bound}\,\widetilde{\mathcal{L}}(f) \leqslant (\pi_n g)_{\mathbf{1}}$ and $(\pi_n g)_{\mathbf{1}} \leqslant \text{E-bound}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-bound}\,\widetilde{\mathcal{L}}(f) \leqslant (\pi_n g)_{\mathbf{2}}$ and $(\pi_n g)_{\mathbf{2}} \leqslant \text{N-bound}\,\widetilde{\mathcal{L}}(f)$.

We now state several propositions:

(25) Every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ is in the area of f.

(26) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose g is in the area of f. Let given i, j. If $i \in \operatorname{dom} g$ and $j \in \operatorname{dom} g$, then $\operatorname{mid}(g,i,j)$ is in the area of f.

(27) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i, j. If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f,i,j)$ is in the area of f.

(28) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g, h be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose g is in the area of f and h is in the area of f. Then $g \frown h$ is in the area of f.

(29) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\langle \text{NE-corner}\,\widetilde{\mathcal{L}}(f) \rangle$ is in the area of f.

(30) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\langle$NW-corner $\widetilde{\mathcal{L}}(f)\rangle$ is in the area of f.

(31) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\langle$SE-corner $\widetilde{\mathcal{L}}(f)\rangle$ is in the area of f.

(32) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\langle$SW-corner $\widetilde{\mathcal{L}}(f)\rangle$ is in the area of f.

6. Horizontal and vertical connections

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that g is a h.c. for f if and only if:

(Def. 2) g is in the area of f and $(\pi_1 g)_{\mathbf{1}}$ = W-bound $\widetilde{\mathcal{L}}(f)$ and $(\pi_{\mathrm{len}\, g} g)_{\mathbf{1}}$ = E-bound $\widetilde{\mathcal{L}}(f)$.

We say that g is a v.c. for f if and only if:

(Def. 3) g is in the area of f and $(\pi_1 g)_{\mathbf{2}}$ = S-bound $\widetilde{\mathcal{L}}(f)$ and $(\pi_{\mathrm{len}\, g} g)_{\mathbf{2}}$ = N-bound $\widetilde{\mathcal{L}}(f)$.

Next we state the proposition

(33) Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and g, h be S-sequences in $\mathbb{R}^2$. If g is a h.c. for f and h is a v.c. for f, then $\widetilde{\mathcal{L}}(g)$ meets $\widetilde{\mathcal{L}}(h)$.

7. Orientation

Let f be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that f is clockwise oriented if and only if:

(Def. 4) $\pi_2 f_{\circlearrowleft}^{\text{N-min}\,\widetilde{\mathcal{L}}(f)} \in$ N-most $\widetilde{\mathcal{L}}(f)$.

The following proposition is true

(34) Let f be a non constant standard special circular sequence. If $\pi_1 f$ = N-min $\widetilde{\mathcal{L}}(f)$, then f is clockwise oriented iff $\pi_2 f \in$ N-most $\widetilde{\mathcal{L}}(f)$.

Let us note that $\square_{\mathcal{E}^2}$ is compact.

We now state several propositions:

(35) N-bound $\square_{\mathcal{E}^2} = 1$.

(36) W-bound $\square_{\mathcal{E}^2} = 0$.

(37) E-bound $\square_{\mathcal{E}^2} = 1$.

(38) S-bound $\square_{\mathcal{E}^2} = 0$.

(39) N-most $\square_{\mathcal{E}^2} = \mathcal{L}([0,1],[1,1])$.

(40) $\text{N-min}\,\square_{\mathcal{E}^2} = [0,1]$.

Let X be a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify that $\mathrm{SpStSeq}\,X$ is clockwise oriented.

One can verify that there exists a non constant standard special circular sequence which is clockwise oriented.

One can prove the following propositions:

(41) Let f be a non constant standard special circular sequence and given i, j. Suppose $i > j$ but $1 < j$ and $i \leqslant \mathrm{len}\,f$ or $1 \leqslant j$ and $i < \mathrm{len}\,f$. Then $\mathrm{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

(42) Let f be a non constant standard special circular sequence and given i, j. Suppose $i < j$ but $1 < i$ and $j \leqslant \mathrm{len}\,f$ or $1 \leqslant i$ and $j < \mathrm{len}\,f$. Then $\mathrm{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

In the sequel f is a clockwise oriented non constant standard special circular sequence.

One can prove the following propositions:

(43) $\text{N-min}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(44) $\text{N-max}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(45) $\text{S-min}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(46) $\text{S-max}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(47) $\text{W-min}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(48) $\text{W-max}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(49) $\text{E-min}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(50) $\text{E-max}\,\widetilde{\mathcal{L}}(f) \in \mathrm{rng}\,f$.

(51) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\,f$ and $1 < i$ or $n < \mathrm{len}\,f$, then $\widetilde{\mathcal{L}}(\mathrm{mid}(f,i,j))$ misses $\widetilde{\mathcal{L}}(\mathrm{mid}(f,m,n))$.

(52) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\,f$ and $1 < i$ or $n < \mathrm{len}\,f$, then $\widetilde{\mathcal{L}}(\mathrm{mid}(f,i,j))$ misses $\widetilde{\mathcal{L}}(\mathrm{mid}(f,n,m))$.

(53) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\,f$ and $1 < i$ or $n < \mathrm{len}\,f$, then $\widetilde{\mathcal{L}}(\mathrm{mid}(f,j,i))$ misses $\widetilde{\mathcal{L}}(\mathrm{mid}(f,n,m))$.

(54) If $1 \leqslant i$ and $i \leqslant j$ and $j < m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\,f$ and $1 < i$ or $n < \mathrm{len}\,f$, then $\widetilde{\mathcal{L}}(\mathrm{mid}(f,j,i))$ misses $\widetilde{\mathcal{L}}(\mathrm{mid}(f,m,n))$.

(55) $(\text{N-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}} < (\text{N-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(56) $\text{N-min}\,\widetilde{\mathcal{L}}(f) \neq \text{N-max}\,\widetilde{\mathcal{L}}(f)$.

(57) $(\text{E-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{E-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(58) $\text{E-min}\,\widetilde{\mathcal{L}}(f) \neq \text{E-max}\,\widetilde{\mathcal{L}}(f)$.

(59) $(\text{S-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}} < (\text{S-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(60) $\text{S-min}\,\widetilde{\mathcal{L}}(f) \neq \text{S-max}\,\widetilde{\mathcal{L}}(f)$.

(61) $(\text{W-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{W-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(62) W-min $\widetilde{\mathcal{L}}(f) \neq$ W-max $\widetilde{\mathcal{L}}(f)$.

(63) $\mathcal{L}$(NW-corner $\widetilde{\mathcal{L}}(f)$, N-min $\widetilde{\mathcal{L}}(f)$) misses $\mathcal{L}$(N-max $\widetilde{\mathcal{L}}(f)$, NE-corner $\widetilde{\mathcal{L}}(f)$).

(64) Let f be a S-sequence in $\mathbb{R}^2$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \neq \pi_1 f$ but $p_{\mathbf{1}} = (\pi_1 f)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (\pi_1 f)_{\mathbf{2}}$ but $\mathcal{L}(p, \pi_1 f) \cap \widetilde{\mathcal{L}}(f) = \{\pi_1 f\}$. Then $\langle p \rangle \frown f$ is a S-sequence in $\mathbb{R}^2$.

(65) Let f be a S-sequence in $\mathbb{R}^2$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p \neq \pi_{\mathrm{len}\, f} f$ but $p_{\mathbf{1}} = (\pi_{\mathrm{len}\, f} f)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (\pi_{\mathrm{len}\, f} f)_{\mathbf{2}}$ but $\mathcal{L}(p, \pi_{\mathrm{len}\, f} f) \cap \widetilde{\mathcal{L}}(f) = \{\pi_{\mathrm{len}\, f} f\}$. Then $f \frown \langle p \rangle$ is a S-sequence in $\mathbb{R}^2$.

8. Appending corners

We now state several propositions:

(66) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f =$ N-max $\widetilde{\mathcal{L}}(f)$ and N-max $\widetilde{\mathcal{L}}(f) \neq$ NE-corner $\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \frown \langle$NE-corner $\widetilde{\mathcal{L}}(f)\rangle$ is a S-sequence in $\mathbb{R}^2$.

(67) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f =$ E-max $\widetilde{\mathcal{L}}(f)$ and E-max $\widetilde{\mathcal{L}}(f) \neq$ NE-corner $\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \frown \langle$NE-corner $\widetilde{\mathcal{L}}(f)\rangle$ is a S-sequence in $\mathbb{R}^2$.

(68) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f =$ S-max $\widetilde{\mathcal{L}}(f)$ and S-max $\widetilde{\mathcal{L}}(f) \neq$ SE-corner $\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \frown \langle$SE-corner $\widetilde{\mathcal{L}}(f)\rangle$ is a S-sequence in $\mathbb{R}^2$.

(69) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_j f =$ E-max $\widetilde{\mathcal{L}}(f)$ and E-max $\widetilde{\mathcal{L}}(f) \neq$ NE-corner $\widetilde{\mathcal{L}}(f)$. Then $(\mathrm{mid}(f, i, j)) \frown \langle$NE-corner $\widetilde{\mathcal{L}}(f)\rangle$ is a S-sequence in $\mathbb{R}^2$.

(70) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_i f =$ N-min $\widetilde{\mathcal{L}}(f)$ and N-min $\widetilde{\mathcal{L}}(f) \neq$ NW-corner $\widetilde{\mathcal{L}}(f)$. Then $\langle$NW-corner $\widetilde{\mathcal{L}}(f)\rangle \frown \mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

(71) Let given i, j. Suppose $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $\mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$ and $\pi_i f =$ W-min $\widetilde{\mathcal{L}}(f)$ and W-min $\widetilde{\mathcal{L}}(f) \neq$ SW-corner $\widetilde{\mathcal{L}}(f)$. Then $\langle$SW-corner $\widetilde{\mathcal{L}}(f)\rangle \frown \mathrm{mid}(f, i, j)$ is a S-sequence in $\mathbb{R}^2$.

Let f be a non constant standard special circular sequence. One can check that $\widetilde{\mathcal{L}}(f)$ is simple closed curve.

9. The order

We now state a number of propositions:

(72) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(73) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f > 1$.

(74) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-max}\,\widetilde{\mathcal{L}}(f) \neq \text{E-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{N-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{E-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(75) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(76) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{E-min}\,\widetilde{\mathcal{L}}(f) \neq \text{S-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(77) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(78) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{S-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{S-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(79) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$ and $\text{N-min}\,\widetilde{\mathcal{L}}(f) \neq \text{W-max}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < (\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f$.

(80) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \text{len}\, f$.

(81) If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $(\text{W-max}\,\widetilde{\mathcal{L}}(f)) \hookleftarrow f < \text{len}\, f$.

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