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Intermediate Value Theorem and Thickness of Simple Closed Curves

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Summary. Various types of the intermediate value theorem ([25]) are proved. For their special cases, the Bolzano theorem is also proved. Using such a theorem, it is shown that if a curve is a simple closed curve, then it is not horizontally degenerated, neither is it vertically degenerated.

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The articles [29], [33], [28], [16], [1], [27], [34], [6], [7], [4], [8], [32], [22], [35], [11], [10], [24], [2], [5], [31], [17], [3], [12], [13], [14], [15], [18], [19], [21], [26], [23], [30], [9], and [20] provide the notation and terminology for this paper.

1. Intermediate Value Theorems and Bolzano Theorem

For simplicity, we adopt the following convention: a, b, c, d, r_1, r_2, r_3, r, r_4, s_1, s_2 are real numbers, p, q are points of $\mathcal{E}^2_{\rm T}$, P is a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and X, Y, Z are non empty topological spaces.

Next we state a number of propositions:

(1) For all a, b, c holds $c \in [a, b]$ iff $a \leqslant c$ and $c \leqslant b$.

(2) Let f be a continuous mapping from X into Y and g be a continuous mapping from Y into Z. Then $g \cdot f$ is a continuous mapping from X into Z.

(3) Let A, B be subsets of the carrier of X. If A is open and B is open and $A \cap B = \emptyset_X$, then A and B are separated.

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(4) Let A, B_1, B_2 be subsets of the carrier of X. Suppose B_1 is open and B_2 is open and $B_1 \cap A \neq \emptyset$ and $B_2 \cap A \neq \emptyset$ and $A \subseteq B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. Then A is not connected.

(5) Let f be a continuous mapping from X into Y and A be a subset of the carrier of X. If A is connected and $A \neq \emptyset$, then $f^{\circ}A$ is connected.

(6) For all r_1, r_2 such that $r_1 \leqslant r_2$ holds $\Omega_{[(r_1),\, r_2]_{\rm T}}$ is connected.

(7) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a < r\}$ holds A is open.

(8) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a > r\}$ holds A is open.

(9) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and given a. Suppose $a \notin A$ and there exist b, d such that $b \in A$ and $d \in A$ and $b < a$ and $a < d$. Then A is not connected.

(10) Let X be a non empty topological space, x_1, x_2 be points of X, a, b, d be real numbers, and f be a continuous mapping from X into $\mathbb{R}^{\mathbf{1}}$. Suppose X is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leqslant d$ and $d \leqslant b$. Then there exists a point x_3 of X such that $f(x_3) = d$.

(11) Let X be a non empty topological space, x_1, x_2 be points of X, B be a subset of the carrier of X, a, b, d be real numbers, and f be a continuous mapping from X into $\mathbb{R}^{\mathbf{1}}$. Suppose B is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leqslant d$ and $d \leqslant b$ and $x_1 \in B$ and $x_2 \in B$. Then there exists a point x_3 of X such that $x_3 \in B$ and $f(x_3) = d$.

(12) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous mapping from $[(r_1),\, r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a < d$ and $d < b$. Then there exists r_3 such that $f(r_3) = d$ and $r_1 < r_3$ and $r_3 < r_2$.

(13) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous mapping from $[(r_1),\, r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a > d$ and $d > b$. Then there exists r_3 such that $f(r_3) = d$ and $r_1 < r_3$ and $r_3 < r_2$.

(14) Let r_1, r_2 be real numbers, g be a continuous mapping from $[(r_1),\, r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$, and given s_1, s_2. Suppose $r_1 < r_2$ and $s_1 \cdot s_2 < 0$ and $s_1 = g(r_1)$ and $s_2 = g(r_2)$. Then there exists r_4 such that $g(r_4) = 0$ and $r_1 < r_4$ and $r_4 < r_2$.

(15) Let g be a map from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ and given s_1, s_2. Suppose g is continuous and $g(0) \neq g(1)$ and $s_1 = g(0)$ and $s_2 = g(1)$. Then there exists r_4 such that $0 < r_4$ and $r_4 < 1$ and $g(r_4) = \frac{s_1+s_2}{2}$.

2. Simple Closed Curves Are Not Flat

Next we state a number of propositions:

(16) For every map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj1}$ holds f is continuous.

(17) For every map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj2}$ holds f is continuous.

(18) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{1}} = g(r)$.

(19) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{2}} = g(r)$.

(20) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{2}} = r$.

(21) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{1}} = r$.

(22) For every compact non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a simple closed curve holds N-bound $C >$ S-bound C.

(23) For every compact non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that C is a simple closed curve holds E-bound $C >$ W-bound C.

(24) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is a simple closed curve holds S-min $P \neq$ N-max P.

(25) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that P is a simple closed curve holds W-min $P \neq$ E-max P.

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