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Lebesgue's Covering Lemma, Uniform Continuity and Segmentation of Arcs

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Summary. For mappings from a metric space to a metric space, a notion of uniform continuity is defined. If we introduce natural topologies to the metric spaces, a uniformly continuous function becomes continuous. On the other hand, if the domain is compact, a continuous function is uniformly continuous. For this proof, Lebesgue's covering lemma is also proved. An arc, which is homeomorphic to [0,1], can be devided into small segments, as small as one wishes.

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The notation and terminology used in this paper have been introduced in the following articles: [35], [41], [40], [34], [28], [23], [1], [43], [38], [27], [39], [31], [11], [33], [10], [30], [26], [42], [2], [7], [8], [4], [19], [20], [18], [29], [15], [9], [14], [36], [17], [21], [16], [6], [22], [13], [24], [3], [5], [32], [12], [25], and [37].

1. Lebesgue's Covering Lemma

We adopt the following rules: s, s_1, s_2, t, r, r_1, r_2 are real numbers, n, m are natural numbers, and X, Y are non empty metric spaces.

The following two propositions are true:

(1) $t - r - (t - s) = -r + s$ and $t - r - (t - s) = s - r$.

(2) For every r such that $r > 0$ there exists a natural number n such that $n > 0$ and $\frac{1}{n} < r$.

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Let X, Y be non empty metric structures and let f be a map from X into Y. We say that f is uniformly continuous if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let given r. Suppose $0 < r$. Then there exists s such that $0 < s$ and for all elements u_1, u_2 of the carrier of X such that $\rho(u_1, u_2) < s$ holds $\rho(f_{u_1}, f_{u_2}) < r$.

Next we state several propositions:

(3) Let X be a non empty topological space, M be a metric space, and f be a map from X into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of M, and P be a subset of the carrier of $M_{\rm top}$. If $P = \mathrm{Ball}(u, r)$, then $f^{-1}(P)$ is open.

(4) Let N, M be metric spaces and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose that for every real number r and for every element u of the carrier of N and for every element u_1 of the carrier of M such that $r > 0$ and $u_1 = f(u)$ there exists s such that $s > 0$ and for every element w of the carrier of N and for every element w_1 of the carrier of M such that $w_1 = f(w)$ and $\rho(u, w) < s$ holds $\rho(u_1, w_1) < r$. Then f is continuous.

(5) Let N be a metric space, M be a non empty metric space, and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of N, and u_1 be an element of the carrier of M. Suppose $r > 0$ and $u_1 = f(u)$. Then there exists s such that

(i) $s > 0$, and

(ii) for every element w of the carrier of N and for every element w_1 of the carrier of M such that $w_1 = f(w)$ and $\rho(u, w) < s$ holds $\rho(u_1, w_1) < r$.

(6) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from $N_{\rm top}$ into $M_{\rm top}$. If $f = g$ and f is uniformly continuous, then g is continuous.

(7) Let N be a non empty metric space and G be a family of subsets of $N_{\rm top}$. Suppose G is a cover of $N_{\rm top}$ and open and $N_{\rm top}$ is compact. Then there exists r such that

(i) $r > 0$, and

(ii) for all elements w_1, w_2 of the carrier of N such that $\rho(w_1, w_2) < r$ there exists a subset G_1 of the carrier of $N_{\rm top}$ such that $w_1 \in G_1$ and $w_2 \in G_1$ and $G_1 \in G$.

2. Uniformity of Continuous Functions on Compact Spaces

Next we state three propositions:

(8) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose $g = f$ and $N_{\rm top}$ is compact and g is continuous. Then f is uniformly continuous.

(9) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ and f be a map from $[0, 1]_{\rm M}$ into $\mathcal{E}^n$. If g is continuous and $f = g$, then f is uniformly continuous.

(10) Let P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$, Q be a non empty subset of the carrier of $\mathcal{E}^n$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T}){\restriction}P$, and f be a map from $[0, 1]_{\rm M}$ into $\mathcal{E}^n{\restriction}Q$. If $P = Q$ and g is continuous and $f = g$, then f is uniformly continuous.

3. Segmentation of Arcs

We now state four propositions:

(11) For every map g from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ there exists a map f from $[0, 1]_{\rm M}$ into $\mathcal{E}^n$ such that $f = g$.

(12) For every r such that $r \geqslant 0$ holds $\lceil r \rceil \geqslant 0$ and $\lfloor r \rfloor \geqslant 0$ and $\lceil r \rceil$ is a natural number and $\lfloor r \rfloor$ is a natural number.

(13) For all r, s holds $|r - s| = |s - r|$.

(14) For all r_1, r_2, s_1, s_2 such that $r_1 \in [s_1, s_2]$ and $r_2 \in [s_1, s_2]$ holds $|r_1 - r_2| \leqslant s_2 - s_1$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is decreasing if and only if:

(Def. 2) For all n, m such that $n \in \operatorname{dom} I_1$ and $m \in \operatorname{dom} I_1$ and $n < m$ holds $I_1(n) > I_1(m)$.

We now state the proposition

(15) Let e be a real number, g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$, and p_1, p_2 be elements of $\mathcal{E}^n_{\rm T}$. Suppose $e > 0$ and g is continuous and one-to-one and $g(0) = p_1$ and $g(1) = p_2$. Then there exists a finite sequence h of elements of $\mathbb{R}$ such that

(i) $h(1) = 1$,

(ii) $h(\operatorname{len} h) = 0$,

(iii) $5 \leqslant \operatorname{len} h$,

(iv) $\operatorname{rng} h \subseteq$ the carrier of $\mathbb{I}$,

(v) h is decreasing, and

(vi) for every natural number i and for every subset Q of the carrier of $\mathbb{I}$ and for every subset W of the carrier of $\mathcal{E}^n$ such that $1 \leqslant i$ and $i < \operatorname{len} h$ and $Q = [\pi_{i+1}h, \pi_i h]$ and $W = g^\circ Q$ holds $\emptyset W < e$.

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