FORMALIZED MATHEMATICS
Volume 6, Number 4, 1997
University of Białystok

Algebraic and Arithmetic Lattices. Part II[1]

Robert Milewski
University of Białystok

Summary. The article is a translation of [13, pp. 89–92]

MML Identifier: `WAYBEL15`.

The articles [21], [22], [1], [8], [9], [12], [20], [19], [18], [3], [11], [17], [2], [4], [14], [24], [6], [5], [10], [7], [23], [15], and [16] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) Let R be a relational structure and S be a full relational substructure of R. Then every full relational substructure of S is a full relational substructure of R.

(2) Let X, Y, Z be non empty 1-sorted structures, f be a map from X into Y, and g be a map from Y into Z. If f is onto and g is onto, then $g \cdot f$ is onto.

(3) For every non empty 1-sorted structure X and for every subset Y of the carrier of X holds $(\mathrm{id}_X)^\circ Y = Y$.

(4) For every set X and for every element a of $2^X_\subseteq$ holds $\uparrow a = \{Y; Y$ ranges over subsets of X: $a \subseteq Y\}$.

(5) Let L be an upper-bounded non empty antisymmetric relational structure and a be an element of L. If $\top_L \leqslant a$, then $a = \top_L$.

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(6) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If g is onto and $\langle g, d\rangle$ is Galois, then T and $\operatorname{Im} d$ are isomorphic.

(7) Let L_1, L_2, L_3 be non empty posets, g_1 be a map from L_1 into L_2, g_2 be a map from L_2 into L_3, d_1 be a map from L_2 into L_1, and d_2 be a map from L_3 into L_2. If $\langle g_1, d_1\rangle$ is Galois and $\langle g_2, d_2\rangle$ is Galois, then $\langle g_2 \cdot g_1, d_1 \cdot d_2\rangle$ is Galois.

(8) Let L_1, L_2 be non empty posets, f be a map from L_1 into L_2, and f_1 be a map from L_2 into L_1. Suppose $f_1 = (f$ **qua** function$)^{-1}$ and f is isomorphic. Then $\langle f, f_1\rangle$ is Galois and $\langle f_1, f\rangle$ is Galois.

(9) For every set X holds $2^X_\subseteq$ is arithmetic.

Next we state four propositions:

(10) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. If f is isomorphic, then for every element x of L_1 holds $f^\circ{\downarrow}x = {\downarrow}f(x)$.

(11) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is continuous holds L_2 is continuous.

(12) Let L_1, L_2 be lattices. Suppose L_1 and L_2 are isomorphic and L_1 is lower-bounded and arithmetic. Then L_2 is arithmetic.

(13) Let L_1, L_2, L_3 be non empty posets, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

2. Maps Preserving Sup's and Inf's

One can prove the following propositions:

(14) Let L_1, L_2 be non empty relational structures, f be a map from L_1 into L_2, and X be a subset of $\operatorname{Im} f$. Then $(f_\circ)^\circ X = X$.

(15) Let X be a set and c be a map from $2^X_\subseteq$ into $2^X_\subseteq$. Suppose c is idempotent and directed-sups-preserving. Then $c_\circ$ is directed-sups-preserving.

(16) Let L be a continuous complete lattice and p be a kernel map from L into L. If p is directed-sups-preserving, then $\operatorname{Im} p$ is a continuous lattice.

(17) Let L be a continuous complete lattice and p be a projection map from L into L. If p is directed-sups-preserving, then $\operatorname{Im} p$ is a continuous lattice.

(18) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an arithmetic lower-bounded lattice A such that there exists a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(19) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists an algebraic lower-bounded lattice A such that there exists

a map from A into L which is onto, infs-preserving, and directed-sups-preserving.

(20) Let L be a lower-bounded lattice. Then L is continuous if and only if there exists a set X and there exists a projection map p from $2^X_\subseteq$ into $2^X_\subseteq$ such that p is directed-sups-preserving and L and $\mathrm{Im}\, p$ are isomorphic.

3. Atoms Elements

Next we state two propositions:

(21) For every non empty relational structure L and for every element x of L holds $x \in \mathrm{PRIME}(L^{\mathrm{op}})$ iff x is co-prime.

(22) Let L be a sup-semilattice and a be an element of L. Then a is co-prime if and only if for all elements x, y of L such that $a \leqslant x \sqcup y$ holds $a \leqslant x$ or $a \leqslant y$.

Let L be a non empty relational structure and let a be an element of L. We say that a is an atom if and only if:

(Def. 1) $\bot_L < a$ and for every element b of L such that $\bot_L < b$ and $b \leqslant a$ holds $b = a$.

Let L be a non empty relational structure. The functor $\mathrm{ATOM}(L)$ yielding a subset of L is defined by:

(Def. 2) For every element x of L holds $x \in \mathrm{ATOM}(L)$ iff x is atom.

The following proposition is true

(23) For every Boolean lattice L and for every element a of L holds a is atom iff a is co-prime and $a \neq \bot_L$.

Let L be a Boolean lattice. Observe that every element of L which is atom is also co-prime.

Next we state several propositions:

(24) For every Boolean lattice L holds $\mathrm{ATOM}(L) = \mathrm{PRIME}(L^{\mathrm{op}}) \setminus \{\bot_L\}$.

(25) For every Boolean lattice L and for all elements x, a of L such that a is atom holds $a \leqslant x$ iff $a \not\leqslant \neg x$.

(26) Let L be a complete Boolean lattice, X be a subset of L, and x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y; y$ ranges over elements of L: $y \in X\}$.

(27) Let L be a lower-bounded antisymmetric non empty relational structure with g.l.b.'s and x, y be elements of L. If x is atom and y is atom and $x \neq y$, then $x \sqcap y = \bot_L$.

(28) Let L be a complete Boolean lattice, x be an element of L, and A be a subset of L. If $A \subseteq \mathrm{ATOM}(L)$, then $x \in A$ iff x is atom and $x \leqslant \sup A$.

(29) Let L be a complete Boolean lattice and X, Y be subsets of L. If $X \subseteq \mathrm{ATOM}(L)$ and $Y \subseteq \mathrm{ATOM}(L)$, then $X \subseteq Y$ iff $\sup X \leqslant \sup Y$.

4. More on the Boolean Lattice

One can prove the following propositions:

(30) For every Boolean lattice L holds L is arithmetic iff there exists a set X such that L and $2^X_\subseteq$ are isomorphic.

(31) For every Boolean lattice L holds L is arithmetic iff L is algebraic.

(32) For every Boolean lattice L holds L is arithmetic iff L is continuous.

(33) For every Boolean lattice L holds L is arithmetic iff L is continuous and L^{op} is continuous.

(34) For every Boolean lattice L holds L is arithmetic iff L is completely-distributive.

(35) Let L be a Boolean lattice. Then L is arithmetic if and only if the following conditions are satisfied:

(i) L is complete, and

(ii) for every element x of L there exists a subset X of L such that $X \subseteq \mathrm{ATOM}(L)$ and $x = \sup X$.

References

[1] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.
[2] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[3] Grzegorz Bancerek. Quantales. *Formalized Mathematics*, 5(**1**):85–91, 1996.
[4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[5] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[6] Grzegorz Bancerek. Duality in relation structures. *Formalized Mathematics*, 6(**2**):227–232, 1997.
[7] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[14] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[15] Beata Madras. Irreducible and prime elements. *Formalized Mathematics*, 6(**2**):233–239, 1997.
[16] Robert Milewski. Algebraic lattices. *Formalized Mathematics*, 6(**2**):249–254, 1997.

[17] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Yozo Toda. The formalization of simple graphs. *Formalized Mathematics*, 5(**1**):137–144, 1996.
[20] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[21] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[23] Mariusz Żynel. The equational characterization of continuous lattices. *Formalized Mathematics*, 6(**2**):199–205, 1997.
[24] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.

Received October 29, 1997