

Full Trees

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The articles [13], [12], [6], [17], [1], [15], [11], [5], [7], [10], [8], [18], [2], [19], [14], [16], [3], [4], and [9] provide the terminology and notation for this paper.

1. TREES AND BINARY TREES

One can prove the following propositions:

- (1) For every set D and for every finite sequence p and for every natural number n such that $p \in D^*$ holds $p \upharpoonright \text{Seg } n \in D^*$.
- (2) For every binary tree T holds every element of T is a finite sequence of elements of *Boolean*.

Let T be a binary tree. We see that the element of T is a finite sequence of elements of *Boolean*.

Next we state several propositions:

- (3) For every tree T such that $T = \{0, 1\}^*$ holds T is binary.
- (4) For every tree T such that $T = \{0, 1\}^*$ holds $\text{Leaves}(T) = \emptyset$.
- (5) Let T be a binary tree, n be a natural number, and t be an element of T . If $t \in T\text{-level}(n)$, then t is a tuple of n and *Boolean*.
- (6) For every tree T such that for every element t of T holds $\text{succ } t = \{t \hat{\ } \langle 0 \rangle, t \hat{\ } \langle 1 \rangle\}$ holds $\text{Leaves}(T) = \emptyset$.
- (7) For every tree T such that for every element t of T holds $\text{succ } t = \{t \hat{\ } \langle 0 \rangle, t \hat{\ } \langle 1 \rangle\}$ holds T is binary.
- (8) For every tree T holds $T = \{0, 1\}^*$ iff for every element t of T holds $\text{succ } t = \{t \hat{\ } \langle 0 \rangle, t \hat{\ } \langle 1 \rangle\}$.

In this article we present several logical schemes. The scheme *Decorated-BinTreeEx* deals with a non empty set \mathcal{A} , an element \mathcal{B} of \mathcal{A} , and a ternary predicate \mathcal{P} , and states that:

There exists a binary tree D decorated with elements of \mathcal{A} such that $\text{dom } D = \{0, 1\}^*$ and $D(\varepsilon) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x \hat{\ } \langle 0 \rangle), D(x \hat{\ } \langle 1 \rangle)]$

provided the following requirement is met:

- For every element a of \mathcal{A} there exist elements b, c of \mathcal{A} such that $\mathcal{P}[a, b, c]$.

The scheme *DecoratedBinTreeEx1* deals with a non empty set \mathcal{A} , an element \mathcal{B} of \mathcal{A} , and two binary predicates \mathcal{P}, \mathcal{Q} , and states that:

There exists a binary tree D decorated with elements of \mathcal{A} such that $\text{dom } D = \{0, 1\}^*$ and $D(\varepsilon) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x \hat{\ } \langle 0 \rangle)]$ and $\mathcal{Q}[D(x), D(x \hat{\ } \langle 1 \rangle)]$

provided the following requirements are met:

- For every element a of \mathcal{A} there exists an element b of \mathcal{A} such that $\mathcal{P}[a, b]$, and
- For every element a of \mathcal{A} there exists an element b of \mathcal{A} such that $\mathcal{Q}[a, b]$.

Let T be a binary tree and let n be a non empty natural number. The functor $\text{NumberOnLevel}(n, T)$ yields a function from $T\text{-level}(n)$ into \mathbb{N} and is defined as follows:

- (Def. 1) For every element t of T such that $t \in T\text{-level}(n)$ and for every tuple F of n and *Boolean* such that $F = \text{Rev}(t)$ holds $(\text{NumberOnLevel}(n, T))(t) = \text{Absval}(F) + 1$.

Let T be a binary tree and let n be a non empty natural number. Note that $\text{NumberOnLevel}(n, T)$ is one-to-one.

2. FULL TREES

Let T be a tree. We say that T is full if and only if:

- (Def. 2) $T = \{0, 1\}^*$.

We now state three propositions:

- (9) $\{0, 1\}^*$ is a tree.
- (10) For every tree T such that $T = \{0, 1\}^*$ and for every natural number n holds $\underbrace{\langle 0, \dots, 0 \rangle}_n \in T\text{-level}(n)$.
- (11) Let T be a tree. Suppose $T = \{0, 1\}^*$. Let n be a non empty natural number and y be a tuple of n and *Boolean*. Then $y \in T\text{-level}(n)$.

Let T be a binary tree and let n be a natural number. Observe that $T\text{-level}(n)$ is finite.

One can check that every tree which is full is also binary.

One can verify that there exists a tree which is full.

One can prove the following proposition

- (12) For every full tree T and for every non empty natural number n holds
 $\text{Seg}(\text{the } n\text{-th power of } 2) \subseteq \text{rng NumberOnLevel}(n, T)$.

Let T be a full tree and let n be a non empty natural number. The functor $\text{FinSeqLevel}(n, T)$ yielding a finite sequence of elements of $T\text{-level}(n)$ is defined by:

(Def. 3) $\text{FinSeqLevel}(n, T) = (\text{NumberOnLevel}(n, T))^{-1}$.

Let T be a full tree and let n be a non empty natural number. Note that $\text{FinSeqLevel}(n, T)$ is one-to-one.

Next we state a number of propositions:

- (13) For every full tree T and for every non empty natural number n holds
 $(\text{NumberOnLevel}(n, T))(\underbrace{\langle 0, \dots, 0 \rangle}_n) = 1$.
- (14) Let T be a full tree, n be a non empty natural number, and y be a tuple of n and *Boolean*. If $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$, then $(\text{NumberOnLevel}(n, T))(\neg y) =$ the n -th power of 2.
- (15) For every full tree T and for every non empty natural number n holds
 $(\text{FinSeqLevel}(n, T))(1) = \underbrace{\langle 0, \dots, 0 \rangle}_n$.
- (16) Let T be a full tree, n be a non empty natural number, and y be a tuple of n and *Boolean*. If $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$, then $(\text{FinSeqLevel}(n, T))(\text{the } n\text{-th power of } 2) = \neg y$.
- (17) Let T be a full tree, n be a non empty natural number, and i be a natural number. If $i \in \text{Seg}(\text{the } n\text{-th power of } 2)$, then $(\text{FinSeqLevel}(n, T))(i) = \text{Rev}(n\text{-BinarySequence}(i - '1))$.
- (18) For every full tree T and for every natural number n holds $\overline{\overline{T\text{-level}(n)}}$ = the n -th power of 2.
- (19) For every full tree T and for every non empty natural number n holds $\text{len FinSeqLevel}(n, T) =$ the n -th power of 2.
- (20) For every full tree T and for every non empty natural number n holds $\text{dom FinSeqLevel}(n, T) = \text{Seg}(\text{the } n\text{-th power of } 2)$.
- (21) For every full tree T and for every non empty natural number n holds $\text{rng FinSeqLevel}(n, T) = T\text{-level}(n)$.
- (22) For every full tree T holds $(\text{FinSeqLevel}(1, T))(1) = \langle 0 \rangle$.

- (23) For every full tree T holds $(\text{FinSeqLevel}(1, T))(2) = \langle 1 \rangle$.
- (24) Let T be a full tree and n, i be non empty natural numbers. Suppose $i \leq$ the $(n + 1)$ -th power of 2. Let F be a tuple of n and *Boolean*. If $F = (\text{FinSeqLevel}(n, T))((i + 1) \div 2)$, then $(\text{FinSeqLevel}(n + 1, T))(i) = F \wedge \langle (i + 1) \bmod 2 \rangle$.

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