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First-countable, Sequential, and Frechet Spaces

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Summary. This article contains a definition of three classes of topological spaces: first-countable, Frechet, and sequential. Next there are some facts about them, that every first-countable space is Frechet and every Frechet space is sequential. Next section constains a formalized construction of topological space which is Frechet but not first-countable. This article is based on [9, pp. 73–81].

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The notation and terminology used here are introduced in the following papers: [19], [2], [15], [4], [5], [6], [11], [1], [13], [3], [12], [14], [10], [20], [21], [18], [16], [8], [7], and [17].

1. Preliminaries

One can prove the following proposition

(1) For every non empty 1-sorted structure T and for every sequence S of T holds $\operatorname{rng} S$ is a subset of T.

Let T be a non empty 1-sorted structure and let S be a sequence of T. Then $\operatorname{rng} S$ is a subset of T.

The following propositions are true:

(2) Let T_1 be a non empty 1-sorted structure, T_2 be a 1-sorted structure, and S be a sequence of T_1. If $\operatorname{rng} S \subseteq$ the carrier of T_2, then S is a sequence of T_2.

(3) For every non empty topological space T and for every point x of T and for every basis B of x holds $B \neq \emptyset$.

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Let T be a non empty topological space and let x be a point of T. Note that every basis of x is non empty.

We now state a number of propositions:

(4) For every topological space T and for all subsets A, B of T such that A is open and B is closed holds $A \setminus B$ is open.

(5) Let T be a topological structure. Suppose that

(i) $\emptyset_T$ is closed,

(ii) Ω_T is closed,

(iii) for all subsets A, B of T such that A is closed and B is closed holds $A \cup B$ is closed, and

(iv) for every family F of subsets of T such that F is closed holds $\bigcap F$ is closed.

Then T is a topological space.

(6) Let T be a topological space, S be a non empty topological structure, and f be a map from T into S. Suppose that for every subset A of S holds A is closed iff $f^{-1}(A)$ is closed. Then S is a topological space.

(7) Let x be a point of the metric space of real numbers and x', r be real numbers. If $x' = x$ and $r > 0$, then $\mathrm{Ball}(x, r) =]x' - r, x' + r[$.

(8) Let A be a subset of $\mathbb{R}^{\mathbf{1}}$. Then A is open if and only if for every real number x such that $x \in A$ there exists a real number r such that $r > 0$ and $]x - r, x + r[\subseteq A$.

(9) For every sequence S of $\mathbb{R}^{\mathbf{1}}$ such that for every natural number n holds $S(n) \in]n - \frac{1}{4}, n + \frac{1}{4}[$ holds $\mathrm{rng}\, S$ is closed.

(10) For every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $B = \mathbb{N}$ holds B is closed.

(11) Let M be a metric space, x be a point of M_{top}, and x' be a point of M. Suppose $x = x'$. Then there exists a basis B of x such that

(i) $B = \{\mathrm{Ball}(x', \frac{1}{n}); n$ ranges over natural numbers: $n \neq 0\}$,

(ii) B is countable, and

(iii) there exists a function f from $\mathbb{N}$ into B such that for every set n such that $n \in \mathbb{N}$ there exists a natural number n' such that $n = n'$ and $f(n) = \mathrm{Ball}(x', \frac{1}{n'+1})$.

(12) For all functions f, g holds $\mathrm{rng}(f {+\cdot} g) = f^{\circ}(\mathrm{dom}\, f \setminus \mathrm{dom}\, g) \cup \mathrm{rng}\, g$.

(13) For all sets A, B such that $B \subseteq A$ holds $(\mathrm{id}_A)^{\circ} B = B$.

(14) For all sets B, x holds $\mathrm{dom}(B \longmapsto x) = B$.

(15) For all sets A, B, x holds $\mathrm{dom}(\mathrm{id}_A {+\cdot} (B \longmapsto x)) = A \cup B$.

(16) For all sets A, B, x such that $B \neq \emptyset$ holds $\mathrm{rng}(\mathrm{id}_A {+\cdot} (B \longmapsto x)) = (A \setminus B) \cup \{x\}$.

(17) For all sets A, B, C, x such that $C \subseteq A$ holds $(\mathrm{id}_A {+\cdot} (B \longmapsto x))^{-1}(C \setminus \{x\}) = C \setminus B \setminus \{x\}$.

(18) For all sets A, B, x such that $x \notin A$ holds $(\mathrm{id}_A {+\cdot} (B \longmapsto x))^{-1}(\{x\}) = B$.

(19) For all sets A, B, C, x such that $C \subseteq A$ and $x \notin A$ holds $(\mathrm{id}_A +\cdot (B \longmapsto x))^{-1}(C \cup \{x\}) = C \cup B$.

(20) For all sets A, B, C, x such that $C \subseteq A$ and $x \notin A$ holds $(\mathrm{id}_A +\cdot (B \longmapsto x))^{-1}(C \setminus \{x\}) = C \setminus B$.

2. First-countable, Sequential, and Frechet Spaces

Let T be a non empty topological structure. We say that T is first-countable if and only if:

(Def. 1) For every point x of T holds there exists a basis of x which is countable.

The following two propositions are true:

(21) For every metric space M holds M_{top} is first-countable.

(22) $\mathbb{R}^{\mathbf{1}}$ is first-countable.

Let us note that $\mathbb{R}^{\mathbf{1}}$ is first-countable.

Let T be a topological structure, let S be a sequence of T, and let x be a point of T. We say that S is convergent to x if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let U_1 be a subset of T. Suppose U_1 is open and $x \in U_1$. Then there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $S(m) \in U_1$.

The following proposition is true

(23) Let T be a non empty topological structure, x be a point of T, and S be a sequence of T. If $S = \mathbb{N} \longmapsto x$, then S is convergent to x.

Let T be a topological structure and let S be a sequence of T. We say that S is convergent if and only if:

(Def. 3) There exists a point x of T such that S is convergent to x.

Let T be a non empty topological structure and let S be a sequence of T. The functor $\mathrm{Lim}\, S$ yields a subset of T and is defined as follows:

(Def. 4) For every point x of T holds $x \in \mathrm{Lim}\, S$ iff S is convergent to x.

Let T be a non empty topological structure. We say that T is Frechet if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let A be a subset of T and x be a point of T. If $x \in \overline{A}$, then there exists a sequence S of T such that $\mathrm{rng}\, S \subseteq A$ and $x \in \mathrm{Lim}\, S$.

Let T be a non empty topological structure. We say that T is sequential if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let A be a subset of T. Then A is closed if and only if for every sequence S of T such that S is convergent and $\mathrm{rng}\, S \subseteq A$ holds $\mathrm{Lim}\, S \subseteq A$.

The following proposition is true

(24) For every non empty topological space T such that T is first-countable holds T is Frechet.

Let us observe that every non empty topological space which is first-countable is also Frechet.

We now state four propositions:

(25) $\mathbb{R}^{\mathbf{1}}$ is Frechet.

(26) Let T be a non empty topological space and A be a subset of T. Suppose A is closed. Let S be a sequence of T. If S is convergent and $\operatorname{rng} S \subseteq A$, then $\operatorname{Lim} S \subseteq A$.

(27) Let T be a non empty topological space. Suppose that for every subset A of T such that for every sequence S of T such that S is convergent and $\operatorname{rng} S \subseteq A$ holds $\operatorname{Lim} S \subseteq A$ holds A is closed. Then T is sequential.

(28) For every non empty topological space T such that T is Frechet holds T is sequential.

Let us mention that every non empty topological space which is Frechet is also sequential.

Next we state the proposition

(29) $\mathbb{R}^{\mathbf{1}}$ is sequential.

3. Counterexample of Frechet but Not First-countable Space

The strict non empty topological space $\mathbb{R}^1{/}_{\mathbb{N}}$ is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of $\mathbb{R}^1{/}_{\mathbb{N}} = (\mathbb{R} \setminus \mathbb{N}) \cup \{\mathbb{R}\}$, and

(ii) there exists a map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^1{/}_{\mathbb{N}}$ such that $f = \mathrm{id}_{\mathbb{R}} + \cdot (\mathbb{N} \longmapsto \mathbb{R})$ and for every subset A of $\mathbb{R}^1{/}_{\mathbb{N}}$ holds A is closed iff $f^{-1}(A)$ is closed.

We now state several propositions:

(30) $\mathbb{R}$ is a point of $\mathbb{R}^1{/}_{\mathbb{N}}$.

(31) Let A be a subset of $\mathbb{R}^1{/}_{\mathbb{N}}$. Then A is open and $\mathbb{R} \in A$ if and only if there exists a subset O of $\mathbb{R}^{\mathbf{1}}$ such that O is open and $\mathbb{N} \subseteq O$ and $A = (O \setminus \mathbb{N}) \cup \{\mathbb{R}\}$.

(32) For every set A holds A is a subset of $\mathbb{R}^1{/}_{\mathbb{N}}$ and $\mathbb{R} \notin A$ iff A is a subset of $\mathbb{R}^{\mathbf{1}}$ and $\mathbb{N} \cap A = \emptyset$.

(33) Let A be a subset of $\mathbb{R}^{\mathbf{1}}$ and B be a subset of $\mathbb{R}^1{/}_{\mathbb{N}}$. If $A = B$, then $\mathbb{N} \cap A = \emptyset$ and A is open iff $\mathbb{R} \notin B$ and B is open.

(34) For every subset A of $\mathbb{R}^1{/}_{\mathbb{N}}$ such that $A = \{\mathbb{R}\}$ holds A is closed.

(35) $\mathbb{R}^1{}_{/\mathbb{N}}$ is not first-countable.

(36) $\mathbb{R}^1{}_{/\mathbb{N}}$ is Frechet.

(37) It is not true that for every non empty topological space T such that T is Frechet holds T is first-countable.

4. Auxiliary Theorems

Next we state three propositions:

(38) $\frac{1}{4} > 0$ and $\frac{1}{4} < \frac{1}{2}$.

(39) For every real number r there exists a natural number n such that $r < n$.

(40) For every real number r such that $r > 0$ there exists a natural number n such that $\frac{1}{n} < r$ and $n \neq 0$.

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