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The Composition of Functors and Transformations in Alternative Categories

Artur Korniłowicz
University of Białystok

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The articles [5], [6], [2], [8], [7], [3], [1], [4], and [9] provide the notation and terminology for this paper.

1. Preliminaries

One can verify that there exists a non empty category structure which is transitive, associative, and strict and has units.

Let A be a non empty transitive category structure and let B be a non empty category structure with units. One can verify that there exists a functor structure from A to B which is strict, comp-preserving, comp-reversing, precovariant, precontravariant, and feasible.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. Observe that there exists a functor structure from A to B which is strict, comp-preserving, comp-reversing, precovariant, precontravariant, feasible, and id-preserving.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. Observe that there exists a functor from A to B which is strict, feasible, covariant, and contravariant.

Next we state several propositions:

(1) Let C be a category, o_1, o_2, o_3, o_4 be objects of C, a be a morphism from o_1 to o_2, b be a morphism from o_2 to o_3, c be a morphism from o_1 to o_4, and d be a morphism from o_4 to o_3. Suppose $b \cdot a = d \cdot c$ and $a \cdot a^{-1} = \mathrm{id}_{(o_2)}$

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and $d^{-1} \cdot d = \mathrm{id}_{(o_4)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_4 \rangle \neq \emptyset$ and $\langle o_4, o_3 \rangle \neq \emptyset$. Then $c \cdot a^{-1} = d^{-1} \cdot b$.

(2) Let A be a non empty transitive category structure, B, C be non empty category structures with units, F be a feasible precovariant functor structure from A to B, G be a functor structure from B to C, and o, o_1 be objects of A. Then $\text{Morph-Map}_{G \cdot F}(o, o_1) = \text{Morph-Map}_G(F(o), F(o_1)) \cdot \text{Morph-Map}_F(o, o_1)$.

(3) Let A be a non empty transitive category structure, B, C be non empty category structures with units, F be a feasible precontravariant functor structure from A to B, G be a functor structure from B to C, and o, o_1 be objects of A. Then $\text{Morph-Map}_{G \cdot F}(o, o_1) = \text{Morph-Map}_G(F(o_1), F(o)) \cdot \text{Morph-Map}_F(o, o_1)$.

(4) Let A be a non empty transitive category structure, B be a non empty category structure with units, and F be a feasible precovariant functor structure from A to B. Then $\mathrm{id}_B \cdot F =$ the functor structure of F.

(5) Let A be a transitive non empty category structure with units, B be a non empty category structure with units, and F be a feasible precovariant functor structure from A to B. Then $F \cdot \mathrm{id}_A =$ the functor structure of F.

For simplicity, we use the following convention: A denotes a non empty category structure, B, C denote non empty reflexive category structures, F denotes a feasible precovariant functor structure from A to B, G denotes a feasible precovariant functor structure from B to C, M denotes a feasible precontravariant functor structure from A to B, N denotes a feasible precontravariant functor structure from B to C, o_1, o_2 denote objects of A, and m denotes a morphism from o_1 to o_2.

The following four propositions are true:

(6) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(G \cdot F)(m) = G(F(m))$.

(7) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(N \cdot M)(m) = N(M(m))$.

(8) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(N \cdot F)(m) = N(F(m))$.

(9) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(G \cdot M)(m) = G(M(m))$.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precovariant comp-preserving functor structure from A to B, and let G be a feasible precovariant comp-preserving functor structure from B to C. One can check that $G \cdot F$ is comp-preserving.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precontravariant comp-reversing functor structure from A to B, and let G be a feasible precontravariant comp-reversing functor structure from B to C. One can check that $G \cdot F$ is comp-preserving.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precovariant comp-preserving functor structure from A to B, and let G be a feasible precontravariant comp-reversing functor structure from B to C. One can verify that $G \cdot F$ is comp-reversing.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precontravariant comp-reversing functor structure from A to B, and let G be a feasible precovariant comp-preserving functor structure from B to C. One can verify that $G \cdot F$ is comp-reversing.

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a covariant functor from A to B, and let G be a covariant functor from B to C. Then $G \cdot F$ is a strict covariant functor from A to C.

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a contravariant functor from A to B, and let G be a contravariant functor from B to C. Then $G \cdot F$ is a strict covariant functor from A to C.

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a covariant functor from A to B, and let G be a contravariant functor from B to C. Then $G \cdot F$ is a strict contravariant functor from A to C.

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a contravariant functor from A to B, and let G be a covariant functor from B to C. Then $G \cdot F$ is a strict contravariant functor from A to C.

For simplicity, we adopt the following convention: A, B, C, D are transitive non empty category structures with units, F_1, F_2, F_3 are covariant functors from A to B, G_1, G_2, G_3 are covariant functors from B to C, H_1, H_2 are covariant functors from C to D, p is a transformation from F_1 to F_2, p_1 is a transformation from F_2 to F_3, q is a transformation from G_1 to G_2, q_1 is a transformation from G_2 to G_3, and r is a transformation from H_1 to H_2.

The following proposition is true

(10) If F_1 is transformable to F_2 and G_1 is transformable to G_2, then $G_1 \cdot F_1$ is transformable to $G_2 \cdot F_2$.

2. The Composition of Functors with Transformations

Let A, B, C be transitive non empty category structures with units, let F_1, F_2 be covariant functors from A to B, let t be a transformation from F_1 to

F_2, and let G be a covariant functor from B to C. Let us assume that F_1 is transformable to F_2. The functor $G \cdot t$ yields a transformation from $G \cdot F_1$ to $G \cdot F_2$ and is defined as follows:

(Def. 1) For every object o of A holds $(G \cdot t)(o) = G(t[o])$.

Next we state the proposition

(11) For every object o of A such that F_1 is transformable to F_2 holds $(G_1 \cdot p)[o] = G_1(p[o])$.

Let A, B, C be transitive non empty category structures with units, let G_1, G_2 be covariant functors from B to C, let F be a covariant functor from A to B, and let s be a transformation from G_1 to G_2. Let us assume that G_1 is transformable to G_2. The functor $s \cdot F$ yielding a transformation from $G_1 \cdot F$ to $G_2 \cdot F$ is defined by:

(Def. 2) For every object o of A holds $(s \cdot F)(o) = s[F(o)]$.

Next we state a number of propositions:

(12) For every object o of A such that G_1 is transformable to G_2 holds $(q \cdot F_1)[o] = q[F_1(o)]$.

(13) If F_1 is transformable to F_2 and F_2 is transformable to F_3, then $G_1 \cdot (p_1 \circ p) = G_1 \cdot p_1 \circ G_1 \cdot p$.

(14) If G_1 is transformable to G_2 and G_2 is transformable to G_3, then $(q_1 \circ q) \cdot F_1 = q_1 \cdot F_1 \circ q \cdot F_1$.

(15) If H_1 is transformable to H_2, then $(r \cdot G_1) \cdot F_1 = r \cdot (G_1 \cdot F_1)$.

(16) If G_1 is transformable to G_2, then $(H_1 \cdot q) \cdot F_1 = H_1 \cdot (q \cdot F_1)$.

(17) If F_1 is transformable to F_2, then $(H_1 \cdot G_1) \cdot p = H_1 \cdot (G_1 \cdot p)$.

(18) $\mathrm{id}_{(G_1)} \cdot F_1 = \mathrm{id}_{G_1 \cdot F_1}$.

(19) $G_1 \cdot \mathrm{id}_{(F_1)} = \mathrm{id}_{G_1 \cdot F_1}$.

(20) If F_1 is transformable to F_2, then $\mathrm{id}_B \cdot p = p$.

(21) If G_1 is transformable to G_2, then $q \cdot \mathrm{id}_B = q$.

3. The Composition of Transformations

Let A, B, C be transitive non empty category structures with units, let F_1, F_2 be covariant functors from A to B, let G_1, G_2 be covariant functors from B to C, let t be a transformation from F_1 to F_2, and let s be a transformation from G_1 to G_2. The functor $s\,t$ yielding a transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$ is defined as follows:

(Def. 3) $s\,t = s \cdot F_2 \circ G_1 \cdot t$.

The following propositions are true:

(22) Let q be a natural transformation from G_1 to G_2. Suppose F_1 is transformable to F_2 and G_1 is naturally transformable to G_2. Then $q\,p = G_2 \cdot p \circ q \cdot F_1$.

(23) If F_1 is transformable to F_2, then $\mathrm{id}_{\mathrm{id}_B}\ p = p$.

(24) If G_1 is transformable to G_2, then $q\ \mathrm{id}_{\mathrm{id}_B} = q$.

(25) If F_1 is transformable to F_2, then $G_1 \cdot p = \mathrm{id}_{(G_1)}\ p$.

(26) If G_1 is transformable to G_2, then $q \cdot F_1 = q\ \mathrm{id}_{(F_1)}$.

We use the following convention: A, B, C, D are categories, F_1, F_2, F_3 are covariant functors from A to B, and G_1, G_2, G_3 are covariant functors from B to C.

One can prove the following proposition

(27) Let H_1, H_2 be covariant functors from C to D, t be a transformation from F_1 to F_2, s be a transformation from G_1 to G_2, and u be a transformation from H_1 to H_2. Suppose F_1 is transformable to F_2 and G_1 is transformable to G_2 and H_1 is transformable to H_2. Then $(u\,s)\,t = u\,(s\,t)$.

In the sequel t denotes a natural transformation from F_1 to F_2, s denotes a natural transformation from G_1 to G_2, and s_1 denotes a natural transformation from G_2 to G_3.

One can prove the following propositions:

(28) If F_1 is naturally transformable to F_2, then $G_1 \cdot t$ is a natural transformation from $G_1 \cdot F_1$ to $G_1 \cdot F_2$.

(29) If G_1 is naturally transformable to G_2, then $s \cdot F_1$ is a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_1$.

(30) Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2. Then $G_1 \cdot F_1$ is naturally transformable to $G_2 \cdot F_2$ and $s\,t$ is a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$.

(31) Let t be a transformation from F_1 to F_2 and t_1 be a transformation from F_2 to F_3. Suppose that
  (i) F_1 is naturally transformable to F_2,
  (ii) F_2 is naturally transformable to F_3,
  (iii) G_1 is naturally transformable to G_2, and
  (iv) G_2 is naturally transformable to G_3.

  Then $(s_1 \circ s)\,(t_1 \circ t) = s_1\,t_1 \circ s\,t$.

4. Natural Equivalences

One can prove the following proposition

(32) Suppose F_1 is naturally transformable to F_2 and F_2 is transformable to F_1 and for every object a of A holds $t[a]$ is iso. Then

(i) F_2 is naturally transformable to F_1, and
(ii) there exists a natural transformation f from F_2 to F_1 such that for every object a of A holds $f(a) = t[a]^{-1}$ and $f[a]$ is iso.

Let A, B be categories and let F_1, F_2 be covariant functors from A to B. We say that F_1, F_2 are naturally equivalent if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) F_1 is naturally transformable to F_2,
(ii) F_2 is transformable to F_1, and
(iii) there exists a natural transformation t from F_1 to F_2 such that for every object a of A holds $t[a]$ is iso.

Let us notice that the predicate F_1, F_2 are naturally equivalent is reflexive and symmetric.

Let A, B be categories and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1, F_2 are naturally equivalent. A natural transformation from F_1 to F_2 is said to be a natural equivalence of F_1 and F_2 if:

(Def. 5) For every object a of A holds it$[a]$ is iso.

In the sequel e is a natural equivalence of F_1 and F_2, e_1 is a natural equivalence of F_2 and F_3, and f is a natural equivalence of G_1 and G_2.

One can prove the following propositions:

(33) Suppose F_1, F_2 are naturally equivalent and F_2, F_3 are naturally equivalent. Then F_1, F_3 are naturally equivalent.

(34) Suppose F_1, F_2 are naturally equivalent and F_2, F_3 are naturally equivalent. Then $e_1 \circ e$ is a natural equivalence of F_1 and F_3.

(35) Suppose F_1, F_2 are naturally equivalent. Then $G_1 \cdot F_1$, $G_1 \cdot F_2$ are naturally equivalent and $G_1 \cdot e$ is a natural equivalence of $G_1 \cdot F_1$ and $G_1 \cdot F_2$.

(36) Suppose G_1, G_2 are naturally equivalent. Then $G_1 \cdot F_1$, $G_2 \cdot F_1$ are naturally equivalent and $f \cdot F_1$ is a natural equivalence of $G_1 \cdot F_1$ and $G_2 \cdot F_1$.

(37) Suppose F_1, F_2 are naturally equivalent and G_1, G_2 are naturally equivalent. Then $G_1 \cdot F_1$, $G_2 \cdot F_2$ are naturally equivalent and $f\,e$ is a natural equivalence of $G_1 \cdot F_1$ and $G_2 \cdot F_2$.

Let A, B be categories, let F_1, F_2 be covariant functors from A to B, and let e be a natural equivalence of F_1 and F_2. Let us assume that F_1, F_2 are naturally equivalent. The functor e^{-1} yielding a natural equivalence of F_2 and F_1 is defined as follows:

(Def. 6) For every object a of A holds $e^{-1}(a) = e[a]^{-1}$.

The following propositions are true:

(38) For every object o of A such that F_1, F_2 are naturally equivalent holds $e^{-1}[o] = e[o]^{-1}$.

(39) If F_1, F_2 are naturally equivalent, then $e \circ e^{-1} = \mathrm{id}_{(F_2)}$.

(40) If F_1, F_2 are naturally equivalent, then $e^{-1} \circ e = \mathrm{id}_{(F_1)}$.

Let A, B be categories and let F be a covariant functor from A to B. Then id_F is a natural equivalence of F and F.

The following three propositions are true:

(41) If F_1, F_2 are naturally equivalent, then $(e^{-1})^{-1} = e$.

(42) Let k be a natural equivalence of F_1 and F_3. Suppose $k = e_1 \circ e$ and F_1, F_2 are naturally equivalent and F_2, F_3 are naturally equivalent. Then $k^{-1} = e^{-1} \circ {e_1}^{-1}$.

(43) $(\mathrm{id}_{(F_1)})^{-1} = \mathrm{id}_{(F_1)}$.

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