

The Field of Quotients Over an Integral Domain

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Summary. We introduce the field of quotients over an integral domain following the well-known construction using pairs over integral domains. In addition we define ring homomorphisms and prove some basic facts about fields of quotients including their universal property.

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The papers [1], [13], [10], [2], [3], [7], [9], [11], [12], [5], [6], [8], and [4] provide the terminology and notation for this paper.

1. PRELIMINARIES

Let I be a non empty zero structure. The functor $Q(I)$ is a subset of $\{ \}$ the carrier of I , the carrier of I and is defined by:

(Def. 1) For every set u holds $u \in Q(I)$ iff there exist elements a, b of the carrier of I such that $u = \langle a, b \rangle$ and $b \neq 0_I$.

Next we state the proposition

- (1) For every non degenerated non empty multiplicative loop with zero structure I holds $Q(I)$ is non empty.

The following two propositions are true:

- (2) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $Q(I)$. Then $u_2 \neq 0_I$.
- (3) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $Q(I)$. Then u_1 is an element of the carrier of I and u_2 is an element of the carrier of I .

Let I be a non degenerated integral domain-like non empty double loop structure and let u, v be elements of $Q(I)$. The functor $u+v$ yielding an element of $Q(I)$ is defined by:

$$(Def. 2) \quad u + v = \langle u_1 \cdot v_2 + v_1 \cdot u_2, u_2 \cdot v_2 \rangle.$$

Let I be a non degenerated integral domain-like non empty double loop structure and let u, v be elements of $Q(I)$. The functor $u \cdot v$ yielding an element of $Q(I)$ is defined as follows:

$$(Def. 3) \quad u \cdot v = \langle u_1 \cdot v_1, u_2 \cdot v_2 \rangle.$$

The following two propositions are true:

- (4) Let I be a non degenerated integral domain-like associative commutative Abelian add-associative distributive non empty double loop structure and u, v, w be elements of $Q(I)$. Then $u+(v+w) = (u+v)+w$ and $u+v = v+u$.
- (5) Let I be a non degenerated integral domain-like associative commutative Abelian non empty double loop structure and u, v, w be elements of $Q(I)$. Then $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ and $u \cdot v = v \cdot u$.

Let I be a non degenerated integral domain-like associative commutative Abelian add-associative distributive non empty double loop structure and let u, v be elements of $Q(I)$. Let us notice that the functor $u+v$ is commutative.

Let I be a non degenerated integral domain-like associative commutative Abelian non empty double loop structure and let u, v be elements of $Q(I)$. Let us note that the functor $u \cdot v$ is commutative.

Let I be a non degenerated non empty multiplicative loop with zero structure and let u be an element of $Q(I)$. The functor $QClass(u)$ is a subset of $Q(I)$ and is defined as follows:

$$(Def. 4) \quad \text{For every element } z \text{ of } Q(I) \text{ holds } z \in QClass(u) \text{ iff } z_1 \cdot u_2 = z_2 \cdot u_1.$$

The following proposition is true

- (6) Let I be a non degenerated commutative non empty multiplicative loop with zero structure and u be an element of $Q(I)$. Then $u \in QClass(u)$.

Let I be a non degenerated commutative non empty multiplicative loop with zero structure and let u be an element of $Q(I)$. Observe that $QClass(u)$ is non empty.

Let I be a non degenerated non empty multiplicative loop with zero structure. The functor $Quot(I)$ is a family of subsets of $Q(I)$ and is defined by:

$$(Def. 5) \quad \text{For every subset } A \text{ of } Q(I) \text{ holds } A \in Quot(I) \text{ iff there exists an element } u \text{ of } Q(I) \text{ such that } A = QClass(u).$$

Next we state the proposition

- (7) For every non degenerated non empty multiplicative loop with zero structure I holds $Quot(I)$ is non empty.

Next we state two propositions:

- (8) Let I be a non degenerated integral domain-like ring and u, v be elements of $\mathbb{Q}(I)$. If there exists an element w of $\text{Quot}(I)$ such that $u \in w$ and $v \in w$, then $u_1 \cdot v_2 = v_1 \cdot u_2$.
- (9) For every non degenerated integral domain-like ring I and for all elements u, v of $\text{Quot}(I)$ such that $u \cap v \neq \emptyset$ holds $u = v$.

2. DEFINING THE OPERATIONS

Let I be a non degenerated integral domain-like ring and let u, v be elements of $\text{Quot}(I)$. The functor $u +_q v$ yielding an element of $\text{Quot}(I)$ is defined by the condition (Def. 6).

- (Def. 6) Let z be an element of $\mathbb{Q}(I)$. Then $z \in u +_q v$ if and only if there exist elements a, b of $\mathbb{Q}(I)$ such that $a \in u$ and $b \in v$ and $z_1 \cdot (a_2 \cdot b_2) = z_2 \cdot (a_1 \cdot b_2 + b_1 \cdot a_2)$.

Let I be a non degenerated integral domain-like ring and let u, v be elements of $\text{Quot}(I)$. The functor $u \cdot_q v$ yielding an element of $\text{Quot}(I)$ is defined by the condition (Def. 7).

- (Def. 7) Let z be an element of $\mathbb{Q}(I)$. Then $z \in u \cdot_q v$ if and only if there exist elements a, b of $\mathbb{Q}(I)$ such that $a \in u$ and $b \in v$ and $z_1 \cdot (a_2 \cdot b_2) = z_2 \cdot (a_1 \cdot b_1)$.

Next we state the proposition

- (10) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $\mathbb{Q}(I)$. Then $\text{QClass}(u)$ is an element of $\text{Quot}(I)$.

We now state two propositions:

- (11) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathbb{Q}(I)$ holds $\text{QClass}(u) +_q \text{QClass}(v) = \text{QClass}(u + v)$.
- (12) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathbb{Q}(I)$ holds $\text{QClass}(u) \cdot_q \text{QClass}(v) = \text{QClass}(u \cdot v)$.

Let I be a non degenerated integral domain-like ring. The functor $0_q(I)$ yielding an element of $\text{Quot}(I)$ is defined by:

- (Def. 8) For every element z of $\mathbb{Q}(I)$ holds $z \in 0_q(I)$ iff $z_1 = 0_I$.

Let I be a non degenerated integral domain-like ring. The functor $1_q(I)$ yielding an element of $\text{Quot}(I)$ is defined as follows:

- (Def. 9) For every element z of $\mathbb{Q}(I)$ holds $z \in 1_q(I)$ iff $z_1 = z_2$.

Let I be a non degenerated integral domain-like ring and let u be an element of $\text{Quot}(I)$. The functor $-_q u$ yielding an element of $\text{Quot}(I)$ is defined by:

- (Def. 10) For every element z of $\mathbb{Q}(I)$ holds $z \in -_q u$ iff there exists an element a of $\mathbb{Q}(I)$ such that $a \in u$ and $z_1 \cdot a_2 = z_2 \cdot -a_1$.

Let I be a non degenerated integral domain-like ring and let u be an element of $\text{Quot}(I)$. Let us assume that $u \neq 0_q(I)$. The functor u_q^{-1} yields an element of $\text{Quot}(I)$ and is defined by:

- (Def. 11) For every element z of $\text{Q}(I)$ holds $z \in u_q^{-1}$ iff there exists an element a of $\text{Q}(I)$ such that $a \in u$ and $z_1 \cdot a_1 = z_2 \cdot a_2$.

The following propositions are true:

- (13) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\text{Quot}(I)$. Then $u +_q (v +_q w) = (u +_q v) +_q w$ and $u +_q v = v +_q u$.
- (14) For every non degenerated integral domain-like ring I and for every element u of $\text{Quot}(I)$ holds $u +_q 0_q(I) = u$ and $0_q(I) +_q u = u$.
- (15) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\text{Quot}(I)$. Then $u \cdot_q (v \cdot_q w) = (u \cdot_q v) \cdot_q w$ and $u \cdot_q v = v \cdot_q u$.
- (16) For every non degenerated integral domain-like ring I and for every element u of $\text{Quot}(I)$ holds $u \cdot_q 1_q(I) = u$ and $1_q(I) \cdot_q u = u$.
- (17) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\text{Quot}(I)$ holds $(u +_q v) \cdot_q w = (u \cdot_q w) +_q (v \cdot_q w)$.
- (18) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\text{Quot}(I)$ holds $u \cdot_q (v +_q w) = (u \cdot_q v) +_q (u \cdot_q w)$.
- (19) For every non degenerated integral domain-like ring I and for every element u of $\text{Quot}(I)$ holds $u +_q -_q u = 0_q(I)$ and $-_q u +_q u = 0_q(I)$.
- (20) Let I be a non degenerated integral domain-like ring and u be an element of $\text{Quot}(I)$. If $u \neq 0_q(I)$, then $u \cdot_q u_q^{-1} = 1_q(I)$ and $u_q^{-1} \cdot_q u = 1_q(I)$.
- (21) For every non degenerated integral domain-like ring I holds $1_q(I) \neq 0_q(I)$.

Let I be a non degenerated integral domain-like ring. The functor $+_q(I)$ yielding a binary operation on $\text{Quot}(I)$ is defined as follows:

- (Def. 12) For all elements u, v of $\text{Quot}(I)$ holds $(+_q(I))(u, v) = u +_q v$.

Let I be a non degenerated integral domain-like ring. The functor $\cdot_q(I)$ yields a binary operation on $\text{Quot}(I)$ and is defined as follows:

- (Def. 13) For all elements u, v of $\text{Quot}(I)$ holds $(\cdot_q(I))(u, v) = u \cdot_q v$.

Let I be a non degenerated integral domain-like ring. The functor $-_q(I)$ yields a unary operation on $\text{Quot}(I)$ and is defined as follows:

- (Def. 14) For every element u of $\text{Quot}(I)$ holds $(-_q(I))(u) = -_q u$.

Let I be a non degenerated integral domain-like ring. The functor $^{-1}_q(I)$ yields a unary operation on $\text{Quot}(I)$ and is defined as follows:

- (Def. 15) For every element u of $\text{Quot}(I)$ holds $(^{-1}_q(I))(u) = u_q^{-1}$.

We now state a number of propositions:

- (22) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\text{Quot}(I)$ holds $(+_q(I))((+_q(I))(u, v), w) = (+_q(I))(u, (+_q(I))(v, w))$.
- (23) For every non degenerated integral domain-like ring I and for all elements u, v of $\text{Quot}(I)$ holds $(+_q(I))(u, v) = (+_q(I))(v, u)$.
- (24) For every non degenerated integral domain-like ring I and for every element u of $\text{Quot}(I)$ holds $(+_q(I))(u, 0_q(I)) = u$ and $(+_q(I))(0_q(I), u) = u$.
- (25) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\text{Quot}(I)$ holds $(\cdot_q(I))((\cdot_q(I))(u, v), w) = (\cdot_q(I))(u, (\cdot_q(I))(v, w))$.
- (26) For every non degenerated integral domain-like ring I and for all elements u, v of $\text{Quot}(I)$ holds $(\cdot_q(I))(u, v) = (\cdot_q(I))(v, u)$.
- (27) For every non degenerated integral domain-like ring I and for every element u of $\text{Quot}(I)$ holds $(\cdot_q(I))(u, 1_q(I)) = u$ and $(\cdot_q(I))(1_q(I), u) = u$.
- (28) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\text{Quot}(I)$. Then $(\cdot_q(I))((+_q(I))(u, v), w) = (+_q(I))((\cdot_q(I))(u, w), (\cdot_q(I))(v, w))$.
- (29) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\text{Quot}(I)$. Then $(\cdot_q(I))(u, (+_q(I))(v, w)) = (+_q(I))((\cdot_q(I))(u, v), (\cdot_q(I))(u, w))$.
- (30) Let I be a non degenerated integral domain-like ring and u be an element of $\text{Quot}(I)$. Then $(+_q(I))(u, (-_q(I))(u)) = 0_q(I)$ and $(+_q(I))((-_q(I))(u), u) = 0_q(I)$.
- (31) Let I be a non degenerated integral domain-like ring and u be an element of $\text{Quot}(I)$. If $u \neq 0_q(I)$, then $(\cdot_q(I))(u, (_q^{-1}(I))(u)) = 1_q(I)$ and $(\cdot_q(I))((_q^{-1}(I))(u), u) = 1_q(I)$.

3. DEFINING THE FIELD OF QUOTIENTS

Let I be a non degenerated integral domain-like ring. The field of quotients of I yields a strict double loop structure and is defined as follows:

(Def. 16) The field of quotients of $I = \langle \text{Quot}(I), +_q(I), \cdot_q(I), 1_q(I), 0_q(I) \rangle$.

Let I be a non degenerated integral domain-like ring. Observe that the field of quotients of I is non empty.

The following propositions are true:

- (32) Let I be a non degenerated integral domain-like ring. Then
- (i) the carrier of the field of quotients of $I = \text{Quot}(I)$,
 - (ii) the addition of the field of quotients of $I = +_q(I)$,

- (iii) the multiplication of the field of quotients of $I = \cdot_q(I)$,
- (iv) the zero of the field of quotients of $I = 0_q(I)$, and
- (v) the unity of the field of quotients of $I = 1_q(I)$.
- (33) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $(+_q(I))(u, v)$ is an element of the carrier of the field of quotients of I .
- (34) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $(-_q(I))(u)$ is an element of the carrier of the field of quotients of I .
- (35) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $(\cdot_q(I))(u, v)$ is an element of the carrier of the field of quotients of I .
- (36) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $(\bar{\cdot}_q^{-1}(I))(u)$ is an element of the carrier of the field of quotients of I .
- (37) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $u + v = (+_q(I))(u, v)$.

Let I be a non degenerated integral domain-like ring. One can verify that the field of quotients of I is add-associative right zeroed and right complementable.

Next we state a number of propositions:

- (38) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $-u = (-_q(I))(u)$.
- (39) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $u \cdot v = (\cdot_q(I))(u, v)$.
- (40) Let I be a non degenerated integral domain-like ring. Then $1_{\text{the field of quotients of } I} = 1_q(I)$ and $0_{\text{the field of quotients of } I} = 0_q(I)$.
- (41) Let I be a non degenerated integral domain-like ring and u, v, w be elements of the carrier of the field of quotients of I . Then $(u + v) + w = u + (v + w)$.
- (42) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $u + v = v + u$.
- (43) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $u + 0_{\text{the field of quotients of } I} = u$.
- (44) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $u + -u = 0_{\text{the field of quotients of } I}$.
- (45) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Then $1_{\text{the field of quotients of } I} \cdot u = u$.

- (46) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I . Then $u \cdot v = v \cdot u$.
- (47) Let I be a non degenerated integral domain-like ring and u, v, w be elements of the carrier of the field of quotients of I . Then $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.
- (48) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I . Suppose $u \neq 0_{\text{the field of quotients of } I}$. Then there exists an element v of the carrier of the field of quotients of I such that $u \cdot v = 1_{\text{the field of quotients of } I}$.
- (49) Let I be a non degenerated integral domain-like ring. Then the field of quotients of I is an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure.

Let I be a non degenerated integral domain-like ring. Note that the field of quotients of I is Abelian commutative associative left unital distributive field-like and non degenerated.

Next we state the proposition

- (50) Let I be a non degenerated integral domain-like ring and x be an element of the carrier of the field of quotients of I . Suppose $x \neq 0_{\text{the field of quotients of } I}$. Let a be an element of the carrier of I . Suppose $a \neq 0_I$. Let u be an element of $Q(I)$. Suppose $x = \text{QClass}(u)$ and $u = \langle a, 1_I \rangle$. Let v be an element of $Q(I)$. If $v = \langle 1_I, a \rangle$, then $x^{-1} = \text{QClass}(v)$.

Let us observe that every add-associative right zeroed right complementable commutative associative left unital distributive field-like non degenerated non empty double loop structure is integral domain-like and right unital.

One can check that there exists a non empty double loop structure which is add-associative, right zeroed, right complementable, Abelian, commutative, associative, left unital, distributive, field-like, and non degenerated.

Let F be a commutative associative left unital distributive field-like non empty double loop structure and let x, y be elements of the carrier of F . The functor $\frac{x}{y}$ yields an element of the carrier of F and is defined as follows:

(Def. 17) $\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

- (51) Let F be a non degenerated field-like ring and a, b, c, d be elements of the carrier of F . If $b \neq 0_F$ and $d \neq 0_F$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.
- (52) Let F be a non degenerated field-like ring and a, b, c, d be elements of the carrier of F . If $b \neq 0_F$ and $d \neq 0_F$, then $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$.

4. DEFINING RING HOMOMORPHISMS

Let R, S be non empty double loop structures and let f be a map from R into S . We say that f is a ring homomorphism if and only if:

(Def. 21)¹ f is additive, multiplicative, and unity-preserving.

Let R, S be non empty double loop structures. One can verify that every map from R into S which is ring homomorphism is also additive, multiplicative, and unity-preserving and every map from R into S which is additive, multiplicative, and unity-preserving is also a ring homomorphism.

Let R, S be non empty double loop structures and let f be a map from R into S . We say that f is a ring epimorphism if and only if:

(Def. 22) f is a ring homomorphism and $\text{rng } f = \text{the carrier of } S$.

We say that f is a ring monomorphism if and only if:

(Def. 23) f is a ring homomorphism and one-to-one.

We introduce f is an embedding as a synonym of f is a ring monomorphism.

Let R, S be non empty double loop structures and let f be a map from R into S . We say that f is a ring isomorphism if and only if:

(Def. 24) f is a ring monomorphism and a ring epimorphism.

Let R, S be non empty double loop structures. Note that every map from R into S which is ring isomorphism is also a ring monomorphism and a ring epimorphism and every map from R into S which is ring monomorphism and ring epimorphism is also a ring isomorphism.

We now state several propositions:

(53) For all rings R, S and for every map f from R into S such that f is a ring homomorphism holds $f(0_R) = 0_S$.

(54) Let R, S be rings and f be a map from R into S . Suppose f is a ring monomorphism. Let x be an element of the carrier of R . Then $f(x) = 0_S$ if and only if $x = 0_R$.

(55) Let R, S be non degenerated field-like rings and f be a map from R into S . Suppose f is a ring homomorphism. Let x be an element of the carrier of R . If $x \neq 0_R$, then $f(x^{-1}) = f(x)^{-1}$.

(56) Let R, S be non degenerated field-like rings and f be a map from R into S . Suppose f is a ring homomorphism. Let x, y be elements of the carrier of R . If $y \neq 0_R$, then $f(x \cdot y^{-1}) = f(x) \cdot f(y)^{-1}$.

(57) Let R, S, T be rings and f be a map from R into S . Suppose f is a ring homomorphism. Let g be a map from S into T . If g is a ring homomorphism, then $g \cdot f$ is a ring homomorphism.

¹The definitions (Def. 18)–(Def. 20) have been removed.

(58) For every non empty double loop structure R holds id_R is a ring homomorphism.

Let R, S be non empty double loop structures. We say that R is embedded in S if and only if:

(Def. 25) There exists a map from R into S which is a ring monomorphism.

Let R, S be non empty double loop structures. We say that R is ring isomorphic to S if and only if:

(Def. 26) There exists a map from R into S which is a ring isomorphism.

Let us note that the predicate R is ring isomorphic to S is symmetric.

5. SOME FURTHER PROPERTIES

Let I be a non empty zero structure and let x, y be elements of the carrier of I . Let us assume that $y \neq 0_I$. The functor $\text{quotient}(x, y)$ yielding an element of $Q(I)$ is defined as follows:

(Def. 27) $\text{quotient}(x, y) = \langle x, y \rangle$.

Let I be a non degenerated integral domain-like ring. The canonical homomorphism of I into quotient field is a map from I into the field of quotients of I and is defined by the condition (Def. 28).

(Def. 28) Let x be an element of the carrier of I . Then (the canonical homomorphism of I into quotient field)(x) = $QClass(\text{quotient}(x, 1_I))$.

Next we state four propositions:

(59) Let I be a non degenerated integral domain-like ring. Then the canonical homomorphism of I into quotient field is a ring homomorphism.

(60) Let I be a non degenerated integral domain-like ring. Then the canonical homomorphism of I into quotient field is an embedding.

(61) For every non degenerated integral domain-like ring I holds I is embedded in the field of quotients of I .

(62) Let F be a non degenerated field-like integral domain-like ring. Then F is ring isomorphic to the field of quotients of F .

Let I be a non degenerated integral domain-like ring. Note that the field of quotients of I is integral domain-like right unital and right-distributive.

One can prove the following proposition

(63) Let I be a non degenerated integral domain-like ring. Then the field of quotients of the field of quotients of I is ring isomorphic to the field of quotients of I .

Let I be a non empty double loop structure, let F be a non empty double loop structure, and let f be a map from I into F . We say that F is a field of quotients for I via f if and only if the conditions (Def. 29) are satisfied.

- (Def. 29)(i) f is a ring monomorphism, and
- (ii) for every add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure F' and for every map f' from I into F' such that f' is a ring monomorphism there exists a map h from F into F' such that h is a ring homomorphism and $h \cdot f = f'$ and for every map h' from F into F' such that h' is a ring homomorphism and $h' \cdot f = f'$ holds $h' = h$.

Next we state two propositions:

- (64) Let I be a non degenerated integral domain-like ring. Then there exists an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure F and there exists a map f from I into F such that F is a field of quotients for I via f .
- (65) Let I be an integral domain-like ring, F, F' be add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structures, f be a map from I into F , and f' be a map from I into F' . Suppose F is a field of quotients for I via f and F' is a field of quotients for I via f' . Then F is ring isomorphic to F' .

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