FORMALIZED MATHEMATICS
Volume 7, Number 1, 1998
University of Białystok

Scott-Continuous Functions[1]

Adam Grabowski
University of Białystok

Summary. The article is a translation of [7, pp. 112–113].

MML Identifier: `WAYBEL17`.

The articles [6], [2], [12], [1], [14], [8], [11], [15], [13], [4], [5], [10], [9], [3], and [16] provide the terminology and notation for this paper.

1. Preliminaries

Let S be a non empty set and let a, b be elements of S. The functor $a, b, ...$ yields a function from $\mathbb{N}$ into S and is defined by the condition (Def. 1).

(Def. 1) Let i be a natural number. Then

- (i) if there exists a natural number k such that $i = 2 \cdot k$, then $(a, b, ...)(i) = a$, and
- (ii) if it is not true that there exists a natural number k such that $i = 2 \cdot k$, then $(a, b, ...)(i) = b$.

We now state two propositions:

(1) Let S, T be non empty reflexive relational structures, f be a map from S into T, and P be a lower subset of T. If f is monotone, then $f^{-1}(P)$ is lower.

(2) Let S, T be non empty reflexive relational structures, f be a map from S into T, and P be an upper subset of T. If f is monotone, then $f^{-1}(P)$ is upper.

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

Let T be an up-complete lattice and let S be an inaccessible subset of T. Note that $-S$ is directly closed.

Next we state the proposition

(3) Let S, T be reflexive antisymmetric non empty relational structures and f be a map from S into T. If f is directed-sups-preserving, then f is monotone.

Let S, T be reflexive antisymmetric non empty relational structures. Observe that every map from S into T which is directed-sups-preserving is also monotone.

Next we state the proposition

(4) Let S, T be up-complete Scott top-lattices and f be a map from S into T. If f is continuous, then f is monotone.

2. Poset of Continuous Maps

Let S be a set and let T be a reflexive relational structure. One can verify that $S \longmapsto T$ is reflexive-yielding.

Let S be a non empty set and let T be a complete lattice. Observe that T^S is complete.

Let S, T be up-complete Scott top-lattices. The functor $\mathrm{SCMaps}(S,T)$ yields a strict full relational substructure of $\mathrm{MonMaps}(S,T)$ and is defined by:

(Def. 2) For every map f from S into T holds $f \in$ the carrier of $\mathrm{SCMaps}(S,T)$ iff f is continuous.

Let S, T be up-complete Scott top-lattices. Note that $\mathrm{SCMaps}(S,T)$ is non empty.

3. Some Special Nets

Let S be a non empty relational structure and let a, b be elements of the carrier of S. The functor $\mathrm{NetStr}(a,b)$ yields a strict non empty net structure over S and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of $\mathrm{NetStr}(a,b) = \mathbb{N}$,

(ii) the mapping of $\mathrm{NetStr}(a,b) = a, b, \ldots$, and

(iii) for all elements i, j of the carrier of $\mathrm{NetStr}(a,b)$ and for all natural numbers i', j' such that $i = i'$ and $j = j'$ holds $i \leqslant j$ iff $i' \leqslant j'$.

Let S be a non empty relational structure and let a, b be elements of the carrier of S. Note that $\mathrm{NetStr}(a,b)$ is reflexive transitive directed and antisymmetric.

We now state four propositions:

(5) Let S be a non empty relational structure, a, b be elements of the carrier of S, and i be an element of the carrier of $\mathrm{NetStr}(a,b)$. Then $(\mathrm{NetStr}(a,b))(i) = a$ or $(\mathrm{NetStr}(a,b))(i) = b$.

(6) Let S be a non empty relational structure, a, b be elements of the carrier of S, i, j be elements of the carrier of $\mathrm{NetStr}(a,b)$, and i', j' be natural numbers such that $i' = i$ and $j' = i' + 1$ and $j' = j$. Then

(i) if $(\mathrm{NetStr}(a,b))(i) = a$, then $(\mathrm{NetStr}(a,b))(j) = b$, and

(ii) if $(\mathrm{NetStr}(a,b))(i) = b$, then $(\mathrm{NetStr}(a,b))(j) = a$.

(7) For every poset S with g.l.b.'s and for all elements a, b of the carrier of S holds $\liminf \mathrm{NetStr}(a,b) = a \sqcap b$.

(8) Let S, T be posets with g.l.b.'s, a, b be elements of the carrier of S, and f be a map from S into T. Then $\liminf(f \cdot \mathrm{NetStr}(a,b)) = f(a) \sqcap f(b)$.

Let S be a non empty relational structure and let D be a non empty subset of S. The functor $\mathrm{NetStr}(D)$ yielding a strict net structure over S is defined by:

(Def. 4) $\mathrm{NetStr}(D) = \langle D, (\text{the internal relation of } S) \mid^2 D, \mathrm{id}_{\text{the carrier of } S} \restriction D \rangle$.

We now state the proposition

(9) Let S be a non empty reflexive relational structure and D be a non empty subset of S. Then $\mathrm{NetStr}(D) = \mathrm{NetStr}(D, \mathrm{id}_{\text{the carrier of } S} \restriction D)$.

Let S be a non empty reflexive relational structure and let D be a directed non empty subset of S. Note that $\mathrm{NetStr}(D)$ is non empty directed and reflexive.

Let S be a non empty reflexive transitive relational structure and let D be a directed non empty subset of S. One can check that $\mathrm{NetStr}(D)$ is transitive.

Let S be a non empty reflexive relational structure and let D be a directed non empty subset of S. Observe that $\mathrm{NetStr}(D)$ is monotone.

We now state the proposition

(10) For every up-complete lattice S and for every directed non empty subset D of S holds $\liminf \mathrm{NetStr}(D) = \sup D$.

4. Monotone Maps

We now state several propositions:

(11) Let S, T be lattices and f be a map from S into T. If for every net N in S holds $f(\liminf N) \leqslant \liminf(f \cdot N)$, then f is monotone.

(12) Let S, T be continuous lower-bounded lattices and f be a map from S into T. Suppose f is directed-sups-preserving. Let x be an element of S. Then $f(x) = \bigsqcup_T \{f(w); w \text{ ranges over elements of } S\colon w \ll x\}$.

(13) Let S be a lattice, T be an up-complete lower-bounded lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T \{f(w); w \text{ ranges over elements of } S\colon w \ll x\}$. Then f is monotone.

(14) Let S be an up-complete lower-bounded lattice, T be a continuous lower-bounded lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \ll x\}$. Let x be an element of S and y be an element of T. Then $y \ll f(x)$ if and only if there exists an element w of S such that $w \ll x$ and $y \ll f(w)$.

(15) Let S, T be non empty relational structures, D be a subset of S, and f be a map from S into T. Suppose that

(i) sup D exists in S and sup $f^\circ D$ exists in T, or

(ii) S is complete and antisymmetric and T is complete and antisymmetric.

If f is monotone, then $\sup(f^\circ D) \leqslant f(\sup D)$.

(16) Let S, T be non empty reflexive antisymmetric relational structures, D be a directed non empty subset of S, and f be a map from S into T. Suppose sup D exists in S and sup $f^\circ D$ exists in T or S is up-complete and T is up-complete. If f is monotone, then $\sup(f^\circ D) \leqslant f(\sup D)$.

(17) Let S, T be non empty relational structures, D be a subset of S, and f be a map from S into T. Suppose that

(i) inf D exists in S and inf $f^\circ D$ exists in T, or

(ii) S is complete and antisymmetric and T is complete and antisymmetric.

If f is monotone, then $f(\inf D) \leqslant \inf(f^\circ D)$.

(18) Let S, T be up-complete lattices, f be a map from S into T, and N be a monotone non empty net structure over S. If f is monotone, then $f \cdot N$ is monotone.

Let S, T be up-complete lattices, let f be a monotone map from S into T, and let N be a monotone non empty net structure over S. Observe that $f \cdot N$ is monotone.

The following two propositions are true:

(19) Let S, T be up-complete lattices and f be a map from S into T. Suppose that for every net N in S holds $f(\liminf N) \leqslant \liminf(f \cdot N)$. Let D be a directed non empty subset of S. Then $\sup(f^\circ D) = f(\sup D)$.

(20) Let S, T be complete lattices, f be a map from S into T, N be a net in S, j be an element of the carrier of N, and j' be an element of the carrier of $f \cdot N$. Suppose $j' = j$. Suppose f is monotone. Then $f(\bigsqcap_S\{N(k); k$ ranges over elements of the carrier of N: $k \geqslant j\}) \leqslant \bigsqcap_T\{(f \cdot N)(l); l$ ranges over elements of the carrier of $f \cdot N : l \geqslant j'\}$.

5. Necessary and Sufficient Conditions of Scott-continuity

We now state two propositions:

(21) Let S, T be complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every net N in S holds $f(\liminf N) \leqslant \liminf(f \cdot N)$.

(22) Let S, T be complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if f is directed-sups-preserving.

Let S, T be complete Scott top-lattices. Observe that every map from S into T which is continuous is also directed-sups-preserving and every map from S into T which is directed-sups-preserving is also continuous.

One can prove the following propositions:

(23) Let S, T be continuous complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every element x of S and for every element y of T holds $y \ll f(x)$ iff there exists an element w of S such that $w \ll x$ and $y \ll f(w)$.

(24) Let S, T be continuous complete Scott top-lattices and f be a map from S into T. Then f is continuous if and only if for every element x of S holds $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \ll x\}$.

(25) Let S be a lattice, T be a complete lattice, and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \leqslant x \ \wedge \ w$ is compact$\}$. Then f is monotone.

(26) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose that for every element x of S holds $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \leqslant x \ \wedge \ w$ is compact$\}$. Let x be an element of S. Then $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \ll x\}$.

(27) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose S is algebraic and T is algebraic. Then f is continuous if and only if for every element x of S and for every element k of T such that $k \in$ the carrier of CompactSublatt(T) holds $k \leqslant f(x)$ iff there exists an element j of S such that $j \in$ the carrier of CompactSublatt(S) and $j \leqslant x$ and $k \leqslant f(j)$.

(28) Let S, T be complete Scott top-lattices and f be a map from S into T. Suppose S is algebraic and T is algebraic. Then f is continuous if and only if for every element x of S holds $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \leqslant x \ \wedge \ w$ is compact$\}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[3] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[5] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[8] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[9] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[10] Robert Milewski. Algebraic lattices. *Formalized Mathematics*, 6(**2**):249–254, 1997.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Andrzej Trybulec. Natural transformations. Discrete categories. *Formalized Mathematics*, 2(**4**):467–474, 1991.
[13] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[14] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.

Received February 13, 1998