

The Properties of Product of Relational Structures¹

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Summary. This work contains useful facts about the product of relational structures. It continues the formalization of [6].

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The articles [14], [1], [13], [12], [3], [5], [9], [4], [10], [11], [2], [7], and [8] provide the notation and terminology for this paper.

1. ON THE ELEMENTS OF PRODUCT OF RELATIONAL STRUCTURES

Let S, T be non empty upper-bounded relational structures. One can check that $\{S, T\}$ is upper-bounded.

Let S, T be non empty lower-bounded relational structures. Observe that $\{S, T\}$ is lower-bounded.

The following propositions are true:

- (1) Let S, T be non empty relational structures. If $\{S, T\}$ is upper-bounded, then S is upper-bounded and T is upper-bounded.
- (2) Let S, T be non empty relational structures. If $\{S, T\}$ is lower-bounded, then S is lower-bounded and T is lower-bounded.
- (3) For all upper-bounded antisymmetric non empty relational structures S, T holds $\top_{\{S, T\}} = \langle \top_S, \top_T \rangle$.
- (4) For all lower-bounded antisymmetric non empty relational structures S, T holds $\perp_{\{S, T\}} = \langle \perp_S, \perp_T \rangle$.

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- (5) Let S, T be lower-bounded antisymmetric non empty relational structures and D be a subset of $[S, T]$. If $[S, T]$ is complete or $\sup D$ exists in $[S, T]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.
- (6) Let S, T be upper-bounded antisymmetric non empty relational structures and D be a subset of $[S, T]$. If $[S, T]$ is complete or $\inf D$ exists in $[S, T]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.
- (7) Let S, T be non empty relational structures and x, y be elements of $[S, T]$. Then $x \leq \{y\}$ if and only if the following conditions are satisfied:
- (i) $x_1 \leq \{y_1\}$, and
 - (ii) $x_2 \leq \{y_2\}$.
- (8) Let S, T be non empty relational structures and x, y, z be elements of $[S, T]$. Then $x \leq \{y, z\}$ if and only if the following conditions are satisfied:
- (i) $x_1 \leq \{y_1, z_1\}$, and
 - (ii) $x_2 \leq \{y_2, z_2\}$.
- (9) Let S, T be non empty relational structures and x, y be elements of $[S, T]$. Then $x \geq \{y\}$ if and only if the following conditions are satisfied:
- (i) $x_1 \geq \{y_1\}$, and
 - (ii) $x_2 \geq \{y_2\}$.
- (10) Let S, T be non empty relational structures and x, y, z be elements of $[S, T]$. Then $x \geq \{y, z\}$ if and only if the following conditions are satisfied:
- (i) $x_1 \geq \{y_1, z_1\}$, and
 - (ii) $x_2 \geq \{y_2, z_2\}$.
- (11) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[S, T]$. Then $\inf \{x, y\}$ exists in $[S, T]$ if and only if $\inf \{x_1, y_1\}$ exists in S and $\inf \{x_2, y_2\}$ exists in T .
- (12) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[S, T]$. Then $\sup \{x, y\}$ exists in $[S, T]$ if and only if $\sup \{x_1, y_1\}$ exists in S and $\sup \{x_2, y_2\}$ exists in T .
- (13) Let S, T be antisymmetric relational structures with g.l.b.'s and x, y be elements of $[S, T]$. Then $(x \sqcap y)_1 = x_1 \sqcap y_1$ and $(x \sqcap y)_2 = x_2 \sqcap y_2$.
- (14) Let S, T be antisymmetric relational structures with l.u.b.'s and x, y be elements of $[S, T]$. Then $(x \sqcup y)_1 = x_1 \sqcup y_1$ and $(x \sqcup y)_2 = x_2 \sqcup y_2$.
- (15) Let S, T be antisymmetric relational structures with g.l.b.'s, x_1, y_1 be elements of S , and x_2, y_2 be elements of T . Then $\langle x_1 \sqcap y_1, x_2 \sqcap y_2 \rangle = \langle x_1, x_2 \rangle \sqcap \langle y_1, y_2 \rangle$.
- (16) Let S, T be antisymmetric relational structures with l.u.b.'s, x_1, y_1 be elements of S , and x_2, y_2 be elements of T . Then $\langle x_1 \sqcup y_1, x_2 \sqcup y_2 \rangle = \langle x_1, x_2 \rangle \sqcup \langle y_1, y_2 \rangle$.

Let S be an antisymmetric relational structure with l.u.b.'s and g.l.b.'s and let x, y be elements of S . Let us note that the predicate y is a complement of x is symmetric.

One can prove the following propositions:

- (17) Let S, T be bounded antisymmetric relational structures with l.u.b.'s and g.l.b.'s and x, y be elements of $[S, T]$. Then x is a complement of y if and only if x_1 is a complement of y_1 and x_2 is a complement of y_2 .
- (18) Let S, T be antisymmetric up-complete non empty reflexive relational structures, a, c be elements of S , and b, d be elements of T . If $\langle a, b \rangle \ll \langle c, d \rangle$, then $a \ll c$ and $b \ll d$.
- (19) Let S, T be up-complete non empty posets, a, c be elements of S , and b, d be elements of T . Then $\langle a, b \rangle \ll \langle c, d \rangle$ if and only if $a \ll c$ and $b \ll d$.
- (20) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x, y be elements of $[S, T]$. If $x \ll y$, then $x_1 \ll y_1$ and $x_2 \ll y_2$.
- (21) Let S, T be up-complete non empty posets and x, y be elements of $[S, T]$. Then $x \ll y$ if and only if the following conditions are satisfied:
 - (i) $x_1 \ll y_1$, and
 - (ii) $x_2 \ll y_2$.
- (22) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[S, T]$. If x is compact, then x_1 is compact and x_2 is compact.
- (23) Let S, T be up-complete non empty posets and x be an element of $[S, T]$. If x_1 is compact and x_2 is compact, then x is compact.

2. ON THE SUBSETS OF PRODUCT OF RELATIONAL STRUCTURES

The following propositions are true:

- (24) Let S, T be antisymmetric relational structures with g.l.b.'s and X, Y be subsets of $[S, T]$. Then $\pi_1(X \sqcap Y) = \pi_1(X) \sqcap \pi_1(Y)$ and $\pi_2(X \sqcap Y) = \pi_2(X) \sqcap \pi_2(Y)$.
- (25) Let S, T be antisymmetric relational structures with l.u.b.'s and X, Y be subsets of $[S, T]$. Then $\pi_1(X \sqcup Y) = \pi_1(X) \sqcup \pi_1(Y)$ and $\pi_2(X \sqcup Y) = \pi_2(X) \sqcup \pi_2(Y)$.
- (26) For all relational structures S, T and for every subset X of $[S, T]$ holds $\downarrow X \subseteq [\downarrow \pi_1(X), \downarrow \pi_2(X)]$.
- (27) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[\downarrow X, \downarrow Y] = \downarrow [X, Y]$.

- (28) For all relational structures S, T and for every subset X of $[S, T]$ holds $\pi_1(\downarrow X) \subseteq \downarrow \pi_1(X)$ and $\pi_2(\downarrow X) \subseteq \downarrow \pi_2(X)$.
- (29) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[S, T]$. Then $\pi_1(\downarrow X) = \downarrow \pi_1(X)$.
- (30) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[S, T]$. Then $\pi_2(\downarrow X) = \downarrow \pi_2(X)$.
- (31) For all relational structures S, T and for every subset X of $[S, T]$ holds $\uparrow X \subseteq [\uparrow \pi_1(X), \uparrow \pi_2(X)]$.
- (32) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[\uparrow X, \uparrow Y] = \uparrow[X, Y]$.
- (33) For all relational structures S, T and for every subset X of $[S, T]$ holds $\pi_1(\uparrow X) \subseteq \uparrow \pi_1(X)$ and $\pi_2(\uparrow X) \subseteq \uparrow \pi_2(X)$.
- (34) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[S, T]$. Then $\pi_1(\uparrow X) = \uparrow \pi_1(X)$.
- (35) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[S, T]$. Then $\pi_2(\uparrow X) = \uparrow \pi_2(X)$.
- (36) Let S, T be non empty relational structures, s be an element of S , and t be an element of T . Then $[\downarrow s, \downarrow t] = \downarrow\langle s, t \rangle$.
- (37) For all non empty relational structures S, T and for every element x of $[S, T]$ holds $\pi_1(\downarrow x) \subseteq \downarrow(x_1)$ and $\pi_2(\downarrow x) \subseteq \downarrow(x_2)$.
- (38) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[S, T]$. Then $\pi_1(\downarrow x) = \downarrow(x_1)$.
- (39) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[S, T]$. Then $\pi_2(\downarrow x) = \downarrow(x_2)$.
- (40) Let S, T be non empty relational structures, s be an element of S , and t be an element of T . Then $[\uparrow s, \uparrow t] = \uparrow\langle s, t \rangle$.
- (41) For all non empty relational structures S, T and for every element x of $[S, T]$ holds $\pi_1(\uparrow x) \subseteq \uparrow(x_1)$ and $\pi_2(\uparrow x) \subseteq \uparrow(x_2)$.
- (42) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[S, T]$. Then $\pi_1(\uparrow x) = \uparrow(x_1)$.
- (43) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[S, T]$. Then $\pi_2(\uparrow x) = \uparrow(x_2)$.
- (44) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\downarrow s, \downarrow t] = \downarrow\langle s, t \rangle$.
- (45) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[S, T]$. Then $\pi_1(\downarrow x) \subseteq \downarrow(x_1)$ and $\pi_2(\downarrow x) \subseteq \downarrow(x_2)$.
- (46) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[S, T]$. Then $\pi_1(\downarrow x) =$

- $\downarrow(x_1)$.
- (47) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[S, T]$. Then $\pi_2(\downarrow x) = \downarrow(x_2)$.
- (48) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\uparrow s, \uparrow t] = \uparrow\langle s, t \rangle$.
- (49) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[S, T]$. Then $\pi_1(\uparrow x) \subseteq \uparrow(x_1)$ and $\pi_2(\uparrow x) \subseteq \uparrow(x_2)$.
- (50) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\text{compactbelow}(s), \text{compactbelow}(t)] = \text{compactbelow}(\langle s, t \rangle)$.
- (51) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[S, T]$. Then $\pi_1(\text{compactbelow}(x)) \subseteq \text{compactbelow}(x_1)$ and $\pi_2(\text{compactbelow}(x)) \subseteq \text{compactbelow}(x_2)$.
- (52) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[S, T]$. Then $\pi_1(\text{compactbelow}(x)) = \text{compactbelow}(x_1)$.
- (53) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[S, T]$. Then $\pi_2(\text{compactbelow}(x)) = \text{compactbelow}(x_2)$.

Let S be a non empty reflexive relational structure. One can verify that every subset of S which is open is also open.

The following propositions are true:

- (54) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[S, T]$. If X is open, then $\pi_1(X)$ is open and $\pi_2(X)$ is open.
- (55) Let S, T be up-complete non empty posets, X be a subset of S , and Y be a subset of T . If X is open and Y is open, then $[X, Y]$ is open.
- (56) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[S, T]$. If X is inaccessible, then $\pi_1(X)$ is inaccessible and $\pi_2(X)$ is inaccessible.
- (57) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be an upper subset of S , and Y be an upper subset of T . If X is inaccessible and Y is inaccessible, then $[X, Y]$ is inaccessible.
- (58) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S , and Y be a subset of T such that $[X, Y]$ is directly closed. Then
- (i) if $Y \neq \emptyset$, then X is directly closed, and
 - (ii) if $X \neq \emptyset$, then Y is directly closed.

- (59) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S , and Y be a subset of T . Suppose X is directly closed and Y is directly closed. Then $\{X, Y\}$ is directly closed.
- (60) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $\{S, T\}$. If X has the property (S), then $\pi_1(X)$ has the property (S) and $\pi_2(X)$ has the property (S).
- (61) Let S, T be up-complete non empty posets, X be a subset of S , and Y be a subset of T . If X has the property (S) and Y has the property (S), then $\{X, Y\}$ has the property (S).

3. ON THE PRODUCTS OF RELATIONAL STRUCTURES

We now state the proposition

- (62) Let S, T be non empty reflexive relational structures. Suppose the relational structure of $S =$ the relational structure of T and S is inf-complete. Then T is inf-complete.

Let S be an inf-complete non empty reflexive relational structure. Observe that the relational structure of S is inf-complete.

Let S, T be inf-complete non empty reflexive relational structures. Observe that $\{S, T\}$ is inf-complete.

The following proposition is true

- (63) Let S, T be non empty reflexive relational structures. If $\{S, T\}$ is inf-complete, then S is inf-complete and T is inf-complete.

Let S, T be complemented bounded antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. Observe that $\{S, T\}$ is complemented.

Next we state the proposition

- (64) Let S, T be bounded antisymmetric relational structures with g.l.b.'s and l.u.b.'s. If $\{S, T\}$ is complemented, then S is complemented and T is complemented.

Let S, T be distributive antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. Observe that $\{S, T\}$ is distributive.

The following propositions are true:

- (65) Let S be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $\{S, T\}$ is distributive, then S is distributive.
- (66) Let S be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $\{S, T\}$ is distributive, then T is distributive.

Let S, T be meet-continuous semilattices. Observe that $\{S, T\}$ satisfies MC.

We now state the proposition

- (67) For all semilattices S, T such that $\{S, T\}$ is meet-continuous holds S is meet-continuous and T is meet-continuous.

Let S, T be up-complete inf-complete non empty posets satisfying axiom of approximation. Note that $\{S, T\}$ satisfies axiom of approximation.

Let S, T be continuous inf-complete non empty posets. Observe that $\{S, T\}$ is continuous.

Next we state the proposition

- (68) Let S, T be up-complete lower-bounded non empty posets. If $\{S, T\}$ is continuous, then S is continuous and T is continuous.

Let S, T be up-complete lower-bounded sup-semilattices satisfying axiom K. Note that $\{S, T\}$ satisfies axiom K.

Let S, T be complete algebraic lower-bounded sup-semilattices. Note that $\{S, T\}$ is algebraic.

The following proposition is true

- (69) For all lower-bounded non empty posets S, T such that $\{S, T\}$ is algebraic holds S is algebraic and T is algebraic.

Let S, T be arithmetic lower-bounded lattices. Note that $\{S, T\}$ is arithmetic.

Next we state the proposition

- (70) For all lower-bounded lattices S, T such that $\{S, T\}$ is arithmetic holds S is arithmetic and T is arithmetic.

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