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The Properties of Product of Relational Structures[1]

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Summary. This work contains useful facts about the product of relational structures. It continues the formalization of [6].

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The articles [14], [1], [13], [12], [3], [5], [9], [4], [10], [11], [2], [7], and [8] provide the notation and terminology for this paper.

1. On the Elements of Product of Relational Structures

Let S, T be non empty upper-bounded relational structures. One can check that $[\!:S, T:\!]$ is upper-bounded.

Let S, T be non empty lower-bounded relational structures. Observe that $[\!:S, T:\!]$ is lower-bounded.

The following propositions are true:

(1) Let S, T be non empty relational structures. If $[\!:S, T:\!]$ is upper-bounded, then S is upper-bounded and T is upper-bounded.

(2) Let S, T be non empty relational structures. If $[\!:S, T:\!]$ is lower-bounded, then S is lower-bounded and T is lower-bounded.

(3) For all upper-bounded antisymmetric non empty relational structures S, T holds $\top_{[\!:S,T:\!]} = \langle \top_S, \top_T \rangle$.

(4) For all lower-bounded antisymmetric non empty relational structures S, T holds $\perp_{[\!:S,T:\!]} = \langle \perp_S, \perp_T \rangle$.

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(5) Let S, T be lower-bounded antisymmetric non empty relational structures and D be a subset of $[\!: S, T :\!]$. If $[\!: S, T :\!]$ is complete or $\sup D$ exists in $[\!: S, T :\!]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(6) Let S, T be upper-bounded antisymmetric non empty relational structures and D be a subset of $[\!: S, T :\!]$. If $[\!: S, T :\!]$ is complete or $\inf D$ exists in $[\!: S, T :\!]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.

(7) Let S, T be non empty relational structures and x, y be elements of $[\!: S, T :\!]$. Then $x \leqslant \{y\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \leqslant \{y_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \leqslant \{y_{\mathbf{2}}\}$.

(8) Let S, T be non empty relational structures and x, y, z be elements of $[\!: S, T :\!]$. Then $x \leqslant \{y, z\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \leqslant \{y_{\mathbf{1}}, z_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \leqslant \{y_{\mathbf{2}}, z_{\mathbf{2}}\}$.

(9) Let S, T be non empty relational structures and x, y be elements of $[\!: S, T :\!]$. Then $x \geqslant \{y\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \geqslant \{y_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \geqslant \{y_{\mathbf{2}}\}$.

(10) Let S, T be non empty relational structures and x, y, z be elements of $[\!: S, T :\!]$. Then $x \geqslant \{y, z\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \geqslant \{y_{\mathbf{1}}, z_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \geqslant \{y_{\mathbf{2}}, z_{\mathbf{2}}\}$.

(11) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[\!: S, T :\!]$. Then $\inf \{x, y\}$ exists in $[\!: S, T :\!]$ if and only if $\inf \{x_{\mathbf{1}}, y_{\mathbf{1}}\}$ exists in S and $\inf \{x_{\mathbf{2}}, y_{\mathbf{2}}\}$ exists in T.

(12) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[\!: S, T :\!]$. Then $\sup \{x, y\}$ exists in $[\!: S, T :\!]$ if and only if $\sup \{x_{\mathbf{1}}, y_{\mathbf{1}}\}$ exists in S and $\sup \{x_{\mathbf{2}}, y_{\mathbf{2}}\}$ exists in T.

(13) Let S, T be antisymmetric relational structures with g.l.b.'s and x, y be elements of $[\!: S, T :\!]$. Then $(x \sqcap y)_{\mathbf{1}} = x_{\mathbf{1}} \sqcap y_{\mathbf{1}}$ and $(x \sqcap y)_{\mathbf{2}} = x_{\mathbf{2}} \sqcap y_{\mathbf{2}}$.

(14) Let S, T be antisymmetric relational structures with l.u.b.'s and x, y be elements of $[\!: S, T :\!]$. Then $(x \sqcup y)_{\mathbf{1}} = x_{\mathbf{1}} \sqcup y_{\mathbf{1}}$ and $(x \sqcup y)_{\mathbf{2}} = x_{\mathbf{2}} \sqcup y_{\mathbf{2}}$.

(15) Let S, T be antisymmetric relational structures with g.l.b.'s, x_1, y_1 be elements of S, and x_2, y_2 be elements of T. Then $\langle x_1 \sqcap y_1, x_2 \sqcap y_2 \rangle = \langle x_1, x_2 \rangle \sqcap \langle y_1, y_2 \rangle$.

(16) Let S, T be antisymmetric relational structures with l.u.b.'s, x_1, y_1 be elements of S, and x_2, y_2 be elements of T. Then $\langle x_1 \sqcup y_1, x_2 \sqcup y_2 \rangle = \langle x_1, x_2 \rangle \sqcup \langle y_1, y_2 \rangle$.

Let S be an antisymmetric relational structure with l.u.b.'s and g.l.b.'s and let x, y be elements of S. Let us note that the predicate y is a complement of x is symmetric.

One can prove the following propositions:

(17) Let S, T be bounded antisymmetric relational structures with l.u.b.'s and g.l.b.'s and x, y be elements of $[: S, T :]$. Then x is a complement of y if and only if $x_{\mathbf{1}}$ is a complement of $y_{\mathbf{1}}$ and $x_{\mathbf{2}}$ is a complement of $y_{\mathbf{2}}$.

(18) Let S, T be antisymmetric up-complete non empty reflexive relational structures, a, c be elements of S, and b, d be elements of T. If $\langle a, b\rangle \ll \langle c, d\rangle$, then $a \ll c$ and $b \ll d$.

(19) Let S, T be up-complete non empty posets, a, c be elements of S, and b, d be elements of T. Then $\langle a, b\rangle \ll \langle c, d\rangle$ if and only if $a \ll c$ and $b \ll d$.

(20) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x, y be elements of $[: S, T :]$. If $x \ll y$, then $x_{\mathbf{1}} \ll y_{\mathbf{1}}$ and $x_{\mathbf{2}} \ll y_{\mathbf{2}}$.

(21) Let S, T be up-complete non empty posets and x, y be elements of $[: S, T :]$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \ll y_{\mathbf{1}}$, and

(ii) $x_{\mathbf{2}} \ll y_{\mathbf{2}}$.

(22) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[: S, T :]$. If x is compact, then $x_{\mathbf{1}}$ is compact and $x_{\mathbf{2}}$ is compact.

(23) Let S, T be up-complete non empty posets and x be an element of $[: S, T :]$. If $x_{\mathbf{1}}$ is compact and $x_{\mathbf{2}}$ is compact, then x is compact.

2. On the Subsets of Product of Relational Structures

The following propositions are true:

(24) Let S, T be antisymmetric relational structures with g.l.b.'s and X, Y be subsets of $[: S, T :]$. Then $\pi_1(X \sqcap Y) = \pi_1(X) \sqcap \pi_1(Y)$ and $\pi_2(X \sqcap Y) = \pi_2(X) \sqcap \pi_2(Y)$.

(25) Let S, T be antisymmetric relational structures with l.u.b.'s and X, Y be subsets of $[: S, T :]$. Then $\pi_1(X \sqcup Y) = \pi_1(X) \sqcup \pi_1(Y)$ and $\pi_2(X \sqcup Y) = \pi_2(X) \sqcup \pi_2(Y)$.

(26) For all relational structures S, T and for every subset X of $[: S, T :]$ holds $\downarrow X \subseteq [: \downarrow\pi_1(X), \downarrow\pi_2(X) :]$.

(27) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[: \downarrow X, \downarrow Y :] = \downarrow[: X, Y :]$.

(28) For all relational structures S, T and for every subset X of $[\!: S, T :\!]$ holds $\pi_1(\downarrow X) \subseteq \downarrow\pi_1(X)$ and $\pi_2(\downarrow X) \subseteq \downarrow\pi_2(X)$.

(29) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[\!: S, T :\!]$. Then $\pi_1(\downarrow X) = \downarrow\pi_1(X)$.

(30) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[\!: S, T :\!]$. Then $\pi_2(\downarrow X) = \downarrow\pi_2(X)$.

(31) For all relational structures S, T and for every subset X of $[\!: S, T :\!]$ holds $\uparrow X \subseteq [\!: \uparrow\pi_1(X), \uparrow\pi_2(X) :\!]$.

(32) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[\!: \uparrow X, \uparrow Y :\!] = \uparrow[\!: X, Y :\!]$.

(33) For all relational structures S, T and for every subset X of $[\!: S, T :\!]$ holds $\pi_1(\uparrow X) \subseteq \uparrow\pi_1(X)$ and $\pi_2(\uparrow X) \subseteq \uparrow\pi_2(X)$.

(34) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[\!: S, T :\!]$. Then $\pi_1(\uparrow X) = \uparrow\pi_1(X)$.

(35) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[\!: S, T :\!]$. Then $\pi_2(\uparrow X) = \uparrow\pi_2(X)$.

(36) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $[\!: \downarrow s, \downarrow t :\!] = \downarrow\langle s, t\rangle$.

(37) For all non empty relational structures S, T and for every element x of $[\!: S, T :\!]$ holds $\pi_1(\downarrow x) \subseteq \downarrow(x_{\mathbf{1}})$ and $\pi_2(\downarrow x) \subseteq \downarrow(x_{\mathbf{2}})$.

(38) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[\!: S, T :\!]$. Then $\pi_1(\downarrow x) = \downarrow(x_{\mathbf{1}})$.

(39) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[\!: S, T :\!]$. Then $\pi_2(\downarrow x) = \downarrow(x_{\mathbf{2}})$.

(40) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $[\!: \uparrow s, \uparrow t :\!] = \uparrow\langle s, t\rangle$.

(41) For all non empty relational structures S, T and for every element x of $[\!: S, T :\!]$ holds $\pi_1(\uparrow x) \subseteq \uparrow(x_{\mathbf{1}})$ and $\pi_2(\uparrow x) \subseteq \uparrow(x_{\mathbf{2}})$.

(42) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[\!: S, T :\!]$. Then $\pi_1(\uparrow x) = \uparrow(x_{\mathbf{1}})$.

(43) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[\!: S, T :\!]$. Then $\pi_2(\uparrow x) = \uparrow(x_{\mathbf{2}})$.

(44) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\!: \twoheaddownarrow s, \twoheaddownarrow t :\!] = \twoheaddownarrow\langle s, t\rangle$.

(45) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[\!: S, T :\!]$. Then $\pi_1(\twoheaddownarrow x) \subseteq \twoheaddownarrow(x_{\mathbf{1}})$ and $\pi_2(\twoheaddownarrow x) \subseteq \twoheaddownarrow(x_{\mathbf{2}})$.

(46) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[\!: S, T :\!]$. Then $\pi_1(\twoheaddownarrow x) =$

$\downarrow(x_{\mathbf{1}})$.

(47) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[\!:S, T:\!]$. Then $\pi_2(\downarrow x) = \downarrow(x_{\mathbf{2}})$.

(48) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\!:\uparrow s, \uparrow t:\!] = \uparrow\langle s, t\rangle$.

(49) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[\!:S, T:\!]$. Then $\pi_1(\uparrow x) \subseteq \uparrow(x_{\mathbf{1}})$ and $\pi_2(\uparrow x) \subseteq \uparrow(x_{\mathbf{2}})$.

(50) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[\!:\mathrm{compactbelow}(s), \mathrm{compactbelow}(t):\!] = \mathrm{compactbelow}(\langle s, t\rangle)$.

(51) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[\!:S, T:\!]$. Then $\pi_1(\mathrm{compactbelow}(x)) \subseteq \mathrm{compactbelow}(x_{\mathbf{1}})$ and $\pi_2(\mathrm{compactbelow}(x)) \subseteq \mathrm{compactbelow}(x_{\mathbf{2}})$.

(52) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[\!:S, T:\!]$. Then $\pi_1(\mathrm{compactbelow}(x)) = \mathrm{compactbelow}(x_{\mathbf{1}})$.

(53) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[\!:S, T:\!]$. Then $\pi_2(\mathrm{compactbelow}(x)) = \mathrm{compactbelow}(x_{\mathbf{2}})$.

Let S be a non empty reflexive relational structure. One can verify that every subset of S which is empty is also open.

The following propositions are true:

(54) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[\!:S, T:\!]$. If X is open, then $\pi_1(X)$ is open and $\pi_2(X)$ is open.

(55) Let S, T be up-complete non empty posets, X be a subset of S, and Y be a subset of T. If X is open and Y is open, then $[\!:X, Y:\!]$ is open.

(56) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[\!:S, T:\!]$. If X is inaccessible, then $\pi_1(X)$ is inaccessible and $\pi_2(X)$ is inaccessible.

(57) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be an upper subset of S, and Y be an upper subset of T. If X is inaccessible and Y is inaccessible, then $[\!:X, Y:\!]$ is inaccessible.

(58) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S, and Y be a subset of T such that $[\!:X, Y:\!]$ is directly closed. Then
  (i) if $Y \neq \emptyset$, then X is directly closed, and
  (ii) if $X \neq \emptyset$, then Y is directly closed.

(59) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S, and Y be a subset of T. Suppose X is directly closed and Y is directly closed. Then $[: X, Y :]$ is directly closed.

(60) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[: S, T :]$. If X has the property (S), then $\pi_1(X)$ has the property (S) and $\pi_2(X)$ has the property (S).

(61) Let S, T be up-complete non empty posets, X be a subset of S, and Y be a subset of T. If X has the property (S) and Y has the property (S), then $[: X, Y :]$ has the property (S).

3. On the Products of Relational Structures

We now state the proposition

(62) Let S, T be non empty reflexive relational structures. Suppose the relational structure of $S =$ the relational structure of T and S is inf-complete. Then T is inf-complete.

Let S be an inf-complete non empty reflexive relational structure. Observe that the relational structure of S is inf-complete.

Let S, T be inf-complete non empty reflexive relational structures. Observe that $[: S, T :]$ is inf-complete.

The following proposition is true

(63) Let S, T be non empty reflexive relational structures. If $[: S, T :]$ is inf-complete, then S is inf-complete and T is inf-complete.

Let S, T be complemented bounded antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. Observe that $[: S, T :]$ is complemented.

Next we state the proposition

(64) Let S, T be bounded antisymmetric relational structures with g.l.b.'s and l.u.b.'s. If $[: S, T :]$ is complemented, then S is complemented and T is complemented.

Let S, T be distributive antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. Observe that $[: S, T :]$ is distributive.

The following propositions are true:

(65) Let S be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $[: S, T :]$ is distributive, then S is distributive.

(66) Let S be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $[: S, T :]$ is distributive, then T is distributive.

Let S, T be meet-continuous semilattices. Observe that $[: S, T :]$ satisfies MC.

We now state the proposition

(67) For all semilattices S, T such that $[: S, T :]$ is meet-continuous holds S is meet-continuous and T is meet-continuous.

Let S, T be up-complete inf-complete non empty posets satisfying axiom of approximation. Note that $[: S, T :]$ satisfies axiom of approximation.

Let S, T be continuous inf-complete non empty posets. Observe that $[: S, T :]$ is continuous.

Next we state the proposition

(68) Let S, T be up-complete lower-bounded non empty posets. If $[: S, T :]$ is continuous, then S is continuous and T is continuous.

Let S, T be up-complete lower-bounded sup-semilattices satisfying axiom K. Note that $[: S, T :]$ satisfies axiom K.

Let S, T be complete algebraic lower-bounded sup-semilattices. Note that $[: S, T :]$ is algebraic.

The following proposition is true

(69) For all lower-bounded non empty posets S, T such that $[: S, T :]$ is algebraic holds S is algebraic and T is algebraic.

Let S, T be arithmetic lower-bounded lattices. Note that $[: S, T :]$ is arithmetic.

Next we state the proposition

(70) For all lower-bounded lattices S, T such that $[: S, T :]$ is arithmetic holds S is arithmetic and T is arithmetic.

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