

On the Characterization of Hausdorff Spaces¹

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The terminology and notation used in this paper are introduced in the following papers: [24], [19], [17], [10], [16], [7], [8], [6], [1], [18], [22], [15], [25], [23], [11], [26], [21], [3], [14], [4], [2], [12], [13], [20], [5], and [9].

1. THE PROPERTIES OF SOME FUNCTIONS

In this paper A, B, X, Y denote sets.

Let X be an empty set. Note that $\bigcup X$ is empty.

Next we state several propositions:

- (1) $(\delta_X)^\circ A \subseteq \{A, A\}$.
- (2) $(\delta_X)^{-1}(\{A, A\}) \subseteq A$.
- (3) For every subset A of X holds $(\delta_X)^{-1}(\{A, A\}) = A$.
- (4) $\text{dom}\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle = \{X, Y\}$ and $\text{rng}\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle = \{Y, X\}$.
- (5) $\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle^\circ\{A, B\} \subseteq \{B, A\}$.
- (6) For every subset A of X and for every subset B of Y holds $\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle^\circ\{A, B\} = \{B, A\}$.
- (7) $\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle$ is one-to-one.

Let X, Y be sets. One can verify that $\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle$ is one-to-one.

The following proposition is true

- (8) $\langle\pi_2(X \times Y), \pi_1(X \times Y)\rangle^{-1} = \langle\pi_2(Y \times X), \pi_1(Y \times X)\rangle$.

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2. THE PROPERTIES OF THE RELATIONAL STRUCTURES

Next we state a number of propositions:

- (9) Let L_1 be a semilattice, L_2 be a non empty relational structure, x, y be elements of L_1 , and x_1, y_1 be elements of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $x = x_1$ and $y = y_1$. Then $x \sqcap y = x_1 \sqcap y_1$.
- (10) Let L_1 be a sup-semilattice, L_2 be a non empty relational structure, x, y be elements of L_1 , and x_1, y_1 be elements of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $x = x_1$ and $y = y_1$. Then $x \sqcup y = x_1 \sqcup y_1$.
- (11) Let L_1 be a semilattice, L_2 be a non empty relational structure, X, Y be subsets of L_1 , and X_1, Y_1 be subsets of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $X = X_1$ and $Y = Y_1$. Then $X \sqcap Y = X_1 \sqcap Y_1$.
- (12) Let L_1 be a sup-semilattice, L_2 be a non empty relational structure, X, Y be subsets of L_1 , and X_1, Y_1 be subsets of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $X = X_1$ and $Y = Y_1$. Then $X \sqcup Y = X_1 \sqcup Y_1$.
- (13) Let L_1 be an antisymmetric up-complete non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, x be an element of L_1 , and y be an element of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $x = y$. Then $\downarrow x = \downarrow y$ and $\uparrow x = \uparrow y$.
- (14) Let L_1 be a meet-continuous semilattice and L_2 be a non empty reflexive relational structure. Suppose the relational structure of $L_1 =$ the relational structure of L_2 . Then L_2 is meet-continuous.
- (15) Let L_1 be a continuous antisymmetric non empty reflexive relational structure and L_2 be a non empty reflexive relational structure. Suppose the relational structure of $L_1 =$ the relational structure of L_2 . Then L_2 is continuous.
- (16) Let L_1, L_2 be relational structures, A be a subset of L_1 , and J be a subset of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $A = J$. Then $\text{sub}(A) = \text{sub}(J)$.
- (17) Let L_1, L_2 be non empty relational structures, A be a relational substructure of L_1 , and J be a relational substructure of L_2 . Suppose that
 - (i) the relational structure of $L_1 =$ the relational structure of L_2 ,
 - (ii) the relational structure of $A =$ the relational structure of J , and
 - (iii) A is meet-inheriting.
 Then J is meet-inheriting.

- (18) Let L_1, L_2 be non empty relational structures, A be a relational substructure of L_1 , and J be a relational substructure of L_2 . Suppose that
- (i) the relational structure of $L_1 =$ the relational structure of L_2 ,
 - (ii) the relational structure of $A =$ the relational structure of J , and
 - (iii) A is join-inheriting.

Then J is join-inheriting.

- (19) Let L_1 be an up-complete antisymmetric non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, X be a subset of L_1 , and Y be a subset of L_2 such that the relational structure of $L_1 =$ the relational structure of L_2 and $X = Y$ and X has the property (S). Then Y has the property (S).

- (20) Let L_1 be an up-complete antisymmetric non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, X be a subset of L_1 , and Y be a subset of L_2 . Suppose the relational structure of $L_1 =$ the relational structure of L_2 and $X = Y$ and X is directly closed. Then Y is directly closed.

- (21) Let N be an antisymmetric relational structure with g.l.b.'s, D, E be subsets of N , and X be an upper subset of N . If $D \cap X = \emptyset$, then $(D \sqcap E) \cap X = \emptyset$.

- (22) Let R be a reflexive non empty relational structure. Then $\Delta_{\text{the carrier of } R} \subseteq (\text{the internal relation of } R) \cap (\text{the internal relation of } R^\sim)$.

- (23) Let R be an antisymmetric relational structure. Then $(\text{the internal relation of } R) \cap (\text{the internal relation of } R^\sim) \subseteq \Delta_{\text{the carrier of } R}$.

- (24) Let R be an upper-bounded semilattice and X be a subset of $\{R, R\}$. If $\inf (\sqcap_R)^\circ X$ exists in R , then \sqcap_R preserves \inf of X .

Let R be a complete semilattice. One can verify that \sqcap_R is infs-preserving.

Next we state the proposition

- (25) Let R be a lower-bounded sup-semilattice and X be a subset of $\{R, R\}$. If $\sup (\sqcup_R)^\circ X$ exists in R , then \sqcup_R preserves \sup of X .

Let R be a complete sup-semilattice. Note that \sqcup_R is sups-preserving.

One can prove the following propositions:

- (26) For every semilattice N and for every subset A of N such that $\text{sub}(A)$ is meet-inheriting holds A is filtered.

- (27) For every sup-semilattice N and for every subset A of N such that $\text{sub}(A)$ is join-inheriting holds A is directed.

- (28) Let N be a transitive relational structure and A, J be subsets of N . If A is coarser than $\uparrow J$, then $\uparrow A \subseteq \uparrow J$.

- (29) For every transitive relational structure N and for all subsets A, J of N such that A is finer than $\downarrow J$ holds $\downarrow A \subseteq \downarrow J$.

- (30) Let N be a non empty reflexive relational structure, x be an element of N , and X be a subset of N . If $x \in X$, then $\uparrow x \subseteq \uparrow X$.
- (31) Let N be a non empty reflexive relational structure, x be an element of N , and X be a subset of N . If $x \in X$, then $\downarrow x \subseteq \downarrow X$.

3. ON THE HAUSDORFF SPACES

In the sequel R, S, T denote non empty topological spaces.

Let T be a non empty topological structure. One can verify that the topological structure of T is non empty.

Let T be a topological space. Observe that the topological structure of T is topological space-like.

Next we state three propositions:

- (32) Let S, T be topological structures and B be a basis of S . Suppose the topological structure of $S =$ the topological structure of T . Then B is a basis of T .
- (33) Let S, T be topological structures and B be a prebasis of S . Suppose the topological structure of $S =$ the topological structure of T . Then B is a prebasis of T .
- (34) Every basis of T is non empty.

Let T be a non empty topological space. Note that every basis of T is non empty.

The following proposition is true

- (35) For every point x of T holds every basis of x is non empty.

Let T be a non empty topological space and let x be a point of T . One can check that every basis of x is non empty.

Next we state a number of propositions:

- (36) Let S_1, T_1, S_2, T_2 be non empty topological spaces, f be a map from S_1 into S_2 , and g be a map from T_1 into T_2 . Suppose that
- (i) the topological structure of $S_1 =$ the topological structure of T_1 ,
 - (ii) the topological structure of $S_2 =$ the topological structure of T_2 ,
 - (iii) $f = g$, and
 - (iv) f is continuous.

Then g is continuous.

- (37) $\Delta_{\text{the carrier of } T} = \{p; p \text{ ranges over points of } [T, T]: \pi_1(\text{the carrier of } T \times \text{the carrier of } T)(p) = \pi_2(\text{the carrier of } T \times \text{the carrier of } T)(p)\}$.
- (38) $\delta_{\text{the carrier of } T}$ is a continuous map from T into $[T, T]$.
- (39) $\pi_1(\text{the carrier of } S \times \text{the carrier of } T)$ is a continuous map from $[S, T]$ into S .

- (40) $\pi_2((\text{the carrier of } S) \times \text{the carrier of } T)$ is a continuous map from $[\![S, T]\!]$ into T .
- (41) Let f be a continuous map from T into S and g be a continuous map from T into R . Then $\langle f, g \rangle$ is a continuous map from T into $[\![S, R]\!]$.
- (42) $\langle \pi_2((\text{the carrier of } S) \times \text{the carrier of } T), \pi_1((\text{the carrier of } S) \times \text{the carrier of } T) \rangle$ is a continuous map from $[\![S, T]\!]$ into $[\![T, S]\!]$.
- (43) Let f be a map from $[\![S, T]\!]$ into $[\![T, S]\!]$. Suppose $f = \langle \pi_2((\text{the carrier of } S) \times \text{the carrier of } T), \pi_1((\text{the carrier of } S) \times \text{the carrier of } T) \rangle$. Then f is a homeomorphism.
- (44) $[\![S, T]\!]$ and $[\![T, S]\!]$ are homeomorphic.
- (45) Let T be a Hausdorff non empty topological space and f, g be continuous maps from S into T . Then
- (i) for every subset X of S such that $X = \{p; p \text{ ranges over points of } S: f(p) \neq g(p)\}$ holds X is open, and
 - (ii) for every subset X of S such that $X = \{p; p \text{ ranges over points of } S: f(p) = g(p)\}$ holds X is closed.
- (46) T is Hausdorff iff for every subset A of $[\![T, T]\!]$ such that $A = \Delta_{\text{the carrier of } T}$ holds A is closed.

Let S, T be topological structures. Note that there exists a refinement of S and T which is strict.

Let S be a non empty topological structure and let T be a topological structure. Observe that there exists a refinement of S and T which is strict and non empty and there exists a refinement of T and S which is strict and non empty.

We now state the proposition

- (47) Let R, S, T be topological structures. Then R is a refinement of S and T if and only if the topological structure of R is a refinement of S and T .

For simplicity, we adopt the following convention: S_1, S_2, T_1, T_2 are non empty topological spaces, R is a refinement of $[\![S_1, T_1]\!]$ and $[\![S_2, T_2]\!]$, R_1 is a refinement of S_1 and S_2 , and R_2 is a refinement of T_1 and T_2 .

The following three propositions are true:

- (48) Suppose the carrier of $S_1 =$ the carrier of S_2 and the carrier of $T_1 =$ the carrier of T_2 . Then $\{[\![U_1, V_1]\!] \cap [\![U_2, V_2]\!]; U_1 \text{ ranges over subsets of } S_1, U_2 \text{ ranges over subsets of } S_2, V_1 \text{ ranges over subsets of } T_1, V_2 \text{ ranges over subsets of } T_2: U_1 \text{ is open} \wedge U_2 \text{ is open} \wedge V_1 \text{ is open} \wedge V_2 \text{ is open}\}$ is a basis of R .
- (49) Suppose the carrier of $S_1 =$ the carrier of S_2 and the carrier of $T_1 =$ the carrier of T_2 . Then the carrier of $[\![R_1, R_2]\!] =$ the carrier of R and the topology of $[\![R_1, R_2]\!] =$ the topology of R .
- (50) Suppose the carrier of $S_1 =$ the carrier of S_2 and the carrier of $T_1 =$ the carrier of T_2 . Then $[\![R_1, R_2]\!]$ is a refinement of $[\![S_1, T_1]\!]$ and $[\![S_2, T_2]\!]$.

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