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On the Characterization of Hausdorff Spaces[1]

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The terminology and notation used in this paper are introduced in the following papers: [24], [19], [17], [10], [16], [7], [8], [6], [1], [18], [22], [15], [25], [23], [11], [26], [21], [3], [14], [4], [2], [12], [13], [20], [5], and [9].

1. The Properties of Some Functions

In this paper A, B, X, Y denote sets.

Let X be an empty set. Note that $\bigcup X$ is empty.

Next we state several propositions:

(1) $(\delta_X)^\circ A \subseteq [: A, A :]$.

(2) $(\delta_X)^{-1}([: A, A :]) \subseteq A$.

(3) For every subset A of X holds $(\delta_X)^{-1}([: A, A :]) = A$.

(4) $\mathrm{dom}\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle = [: X, Y :]$ and $\mathrm{rng}\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle = [: Y, X :]$.

(5) $\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle^\circ [: A, B :] \subseteq [: B, A :]$.

(6) For every subset A of X and for every subset B of Y holds $\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle^\circ [: A, B :] = [: B, A :]$.

(7) $\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle$ is one-to-one.

Let X, Y be sets. One can verify that $\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle$ is one-to-one.

The following proposition is true

(8) $\langle \pi_2(X \times Y), \pi_1(X \times Y)\rangle^{-1} = \langle \pi_2(Y \times X), \pi_1(Y \times X)\rangle$.

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2. The Properties of the Relational Structures

Next we state a number of propositions:

(9) Let L_1 be a semilattice, L_2 be a non empty relational structure, x, y be elements of L_1, and x_1, y_1 be elements of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $x = x_1$ and $y = y_1$. Then $x \sqcap y = x_1 \sqcap y_1$.

(10) Let L_1 be a sup-semilattice, L_2 be a non empty relational structure, x, y be elements of L_1, and x_1, y_1 be elements of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $x = x_1$ and $y = y_1$. Then $x \sqcup y = x_1 \sqcup y_1$.

(11) Let L_1 be a semilattice, L_2 be a non empty relational structure, X, Y be subsets of L_1, and X_1, Y_1 be subsets of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $X = X_1$ and $Y = Y_1$. Then $X \sqcap Y = X_1 \sqcap Y_1$.

(12) Let L_1 be a sup-semilattice, L_2 be a non empty relational structure, X, Y be subsets of L_1, and X_1, Y_1 be subsets of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $X = X_1$ and $Y = Y_1$. Then $X \sqcup Y = X_1 \sqcup Y_1$.

(13) Let L_1 be an antisymmetric up-complete non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, x be an element of L_1, and y be an element of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $x = y$. Then $\twoheaddownarrow x = \twoheaddownarrow y$ and $\twoheaduparrow x = \twoheaduparrow y$.

(14) Let L_1 be a meet-continuous semilattice and L_2 be a non empty reflexive relational structure. Suppose the relational structure of L_1 = the relational structure of L_2. Then L_2 is meet-continuous.

(15) Let L_1 be a continuous antisymmetric non empty reflexive relational structure and L_2 be a non empty reflexive relational structure. Suppose the relational structure of L_1 = the relational structure of L_2. Then L_2 is continuous.

(16) Let L_1, L_2 be relational structures, A be a subset of L_1, and J be a subset of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $A = J$. Then $\mathrm{sub}(A) = \mathrm{sub}(J)$.

(17) Let L_1, L_2 be non empty relational structures, A be a relational substructure of L_1, and J be a relational substructure of L_2. Suppose that

(i) the relational structure of L_1 = the relational structure of L_2,

(ii) the relational structure of A = the relational structure of J, and

(iii) A is meet-inheriting.

Then J is meet-inheriting.

(18) Let L_1, L_2 be non empty relational structures, A be a relational substructure of L_1, and J be a relational substructure of L_2. Suppose that
  (i) the relational structure of L_1 = the relational structure of L_2,
  (ii) the relational structure of A = the relational structure of J, and
  (iii) A is join-inheriting.

Then J is join-inheriting.

(19) Let L_1 be an up-complete antisymmetric non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, X be a subset of L_1, and Y be a subset of L_2 such that the relational structure of L_1 = the relational structure of L_2 and $X = Y$ and X has the property (S). Then Y has the property (S).

(20) Let L_1 be an up-complete antisymmetric non empty reflexive relational structure, L_2 be a non empty reflexive relational structure, X be a subset of L_1, and Y be a subset of L_2. Suppose the relational structure of L_1 = the relational structure of L_2 and $X = Y$ and X is directly closed. Then Y is directly closed.

(21) Let N be an antisymmetric relational structure with g.l.b.'s, D, E be subsets of N, and X be an upper subset of N. If $D \cap X = \emptyset$, then $(D \sqcap E) \cap X = \emptyset$.

(22) Let R be a reflexive non empty relational structure. Then $\triangle_{\text{the carrier of } R} \subseteq$ (the internal relation of R) $\cap$ (the internal relation of $R^{\smile}$).

(23) Let R be an antisymmetric relational structure. Then (the internal relation of R) $\cap$ (the internal relation of $R^{\smile}$) $\subseteq \triangle_{\text{the carrier of } R}$.

(24) Let R be an upper-bounded semilattice and X be a subset of $[\!: R, R :\!]$. If $\inf (\sqcap_R)^\circ X$ exists in R, then $\sqcap_R$ preserves inf of X.

Let R be a complete semilattice. One can verify that $\sqcap_R$ is infs-preserving.

Next we state the proposition

(25) Let R be a lower-bounded sup-semilattice and X be a subset of $[\!: R, R :\!]$. If $\sup (\sqcup_R)^\circ X$ exists in R, then $\sqcup_R$ preserves sup of X.

Let R be a complete sup-semilattice. Note that $\sqcup_R$ is sups-preserving.

One can prove the following propositions:

(26) For every semilattice N and for every subset A of N such that $\mathrm{sub}(A)$ is meet-inheriting holds A is filtered.

(27) For every sup-semilattice N and for every subset A of N such that $\mathrm{sub}(A)$ is join-inheriting holds A is directed.

(28) Let N be a transitive relational structure and A, J be subsets of N. If A is coarser than $\uparrow J$, then $\uparrow A \subseteq \uparrow J$.

(29) For every transitive relational structure N and for all subsets A, J of N such that A is finer than $\downarrow J$ holds $\downarrow A \subseteq \downarrow J$.

(30) Let N be a non empty reflexive relational structure, x be an element of N, and X be a subset of N. If $x \in X$, then $\uparrow x \subseteq \uparrow X$.

(31) Let N be a non empty reflexive relational structure, x be an element of N, and X be a subset of N. If $x \in X$, then $\downarrow x \subseteq \downarrow X$.

3. On the Hausdorff Spaces

In the sequel R, S, T denote non empty topological spaces.

Let T be a non empty topological structure. One can verify that the topological structure of T is non empty.

Let T be a topological space. Observe that the topological structure of T is topological space-like.

Next we state three propositions:

(32) Let S, T be topological structures and B be a basis of S. Suppose the topological structure of S = the topological structure of T. Then B is a basis of T.

(33) Let S, T be topological structures and B be a prebasis of S. Suppose the topological structure of S = the topological structure of T. Then B is a prebasis of T.

(34) Every basis of T is non empty.

Let T be a non empty topological space. Note that every basis of T is non empty.

The following proposition is true

(35) For every point x of T holds every basis of x is non empty.

Let T be a non empty topological space and let x be a point of T. One can check that every basis of x is non empty.

Next we state a number of propositions:

(36) Let S_1, T_1, S_2, T_2 be non empty topological spaces, f be a map from S_1 into S_2, and g be a map from T_1 into T_2. Suppose that

(i) the topological structure of S_1 = the topological structure of T_1,

(ii) the topological structure of S_2 = the topological structure of T_2,

(iii) $f = g$, and

(iv) f is continuous.

Then g is continuous.

(37) $\triangle_{\text{the carrier of } T} = \{p; p$ ranges over points of $[\!: T, T :\!]$: $\pi_1((\text{the carrier of } T) \times \text{the carrier of } T)(p) = \pi_2((\text{the carrier of } T) \times \text{the carrier of } T)(p)\}$.

(38) $\delta_{\text{the carrier of } T}$ is a continuous map from T into $[\!: T, T :\!]$.

(39) $\pi_1((\text{the carrier of } S) \times \text{the carrier of } T\ )$ is a continuous map from $[\!: S, T :\!]$ into S.

(40) π_2((the carrier of S) $\times$ the carrier of T) is a continuous map from $[\!:S, T:\!]$ into T.

(41) Let f be a continuous map from T into S and g be a continuous map from T into R. Then $\langle f, g\rangle$ is a continuous map from T into $[\!:S, R:\!]$.

(42) $\langle\pi_2$((the carrier of S) $\times$ the carrier of T), π_1((the carrier of S) $\times$ the carrier of T)$\rangle$ is a continuous map from $[\!:S, T:\!]$ into $[\!:T, S:\!]$.

(43) Let f be a map from $[\!:S, T:\!]$ into $[\!:T, S:\!]$. Suppose $f = \langle\pi_2$((the carrier of S) $\times$ the carrier of T), π_1((the carrier of S) $\times$ the carrier of T)$\rangle$. Then f is a homeomorphism.

(44) $[\!:S, T:\!]$ and $[\!:T, S:\!]$ are homeomorphic.

(45) Let T be a Hausdorff non empty topological space and f, g be continuous maps from S into T. Then
  (i) for every subset X of S such that $X = \{p; p$ ranges over points of S: $f(p) \neq g(p)\}$ holds X is open, and
  (ii) for every subset X of S such that $X = \{p; p$ ranges over points of S: $f(p) = g(p)\}$ holds X is closed.

(46) T is Hausdorff iff for every subset A of $[\!:T, T:\!]$ such that $A = \triangle_{\text{the carrier of } T}$ holds A is closed.

Let S, T be topological structures. Note that there exists a refinement of S and T which is strict.

Let S be a non empty topological structure and let T be a topological structure. Observe that there exists a refinement of S and T which is strict and non empty and there exists a refinement of T and S which is strict and non empty.

We now state the proposition

(47) Let R, S, T be topological structures. Then R is a refinement of S and T if and only if the topological structure of R is a refinement of S and T.

For simplicity, we adopt the following convention: S_1, S_2, T_1, T_2 are non empty topological spaces, R is a refinement of $[\!:S_1, T_1:\!]$ and $[\!:S_2, T_2:\!]$, R_1 is a refinement of S_1 and S_2, and R_2 is a refinement of T_1 and T_2.

The following three propositions are true:

(48) Suppose the carrier of S_1 = the carrier of S_2 and the carrier of T_1 = the carrier of T_2. Then $\{[\!:U_1, V_1:\!] \cap [\!:U_2, V_2:\!]; U_1$ ranges over subsets of S_1, U_2 ranges over subsets of S_2, V_1 ranges over subsets of T_1, V_2 ranges over subsets of T_2: U_1 is open $\wedge$ U_2 is open $\wedge$ V_1 is open $\wedge$ V_2 is open$\}$ is a basis of R.

(49) Suppose the carrier of S_1 = the carrier of S_2 and the carrier of T_1 = the carrier of T_2. Then the carrier of $[\!:R_1, R_2:\!]$ = the carrier of R and the topology of $[\!:R_1, R_2:\!]$ = the topology of R.

(50) Suppose the carrier of S_1 = the carrier of S_2 and the carrier of T_1 = the carrier of T_2. Then $[\!:R_1, R_2:\!]$ is a refinement of $[\!:S_1, T_1:\!]$ and $[\!:S_2, T_2:\!]$.

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