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Bases and Refinements of Topologies[1]

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The notation and terminology used in this paper are introduced in the following articles: [18], [14], [11], [7], [1], [13], [16], [10], [4], [19], [9], [17], [12], [6], [15], [3], [8], [2], and [5].

1. Subsets as Nets

Let A be a set and let B be a non empty set. Observe that B^A is non empty.

In this article we present several logical schemes. The scheme *FraenkelInvolution* deals with a non empty set $\mathcal{A}$, subsets $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{B} = \{\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{C}\}$

provided the parameters have the following properties:

- $\mathcal{C} = \{\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{B}\}$, and
- For every element a of $\mathcal{A}$ holds $\mathcal{F}(\mathcal{F}(a)) = a$.

The scheme *FraenkelComplement1* deals with a non empty relational structure $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

$\mathcal{B}^{\mathrm{c}} = \{-\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{C}\}$

provided the parameters meet the following requirement:

- $\mathcal{B} = \{\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{C}\}$.

The scheme *FraenkelComplement2* deals with a non empty relational structure $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

$\mathcal{B}^{\mathrm{c}} = \{\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{C}\}$

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provided the parameters meet the following requirement:

- $\mathcal{B} = \{-\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : a \in \mathcal{C}\}$.

We now state several propositions:

(1) For every non empty relational structure R and for all elements x, y of R holds $y \in -\uparrow x$ iff $x \not\leq y$.

(2) Let S be a 1-sorted structure, T be a non empty 1-sorted structure, f be a map from S into T, and X be a subset of the carrier of T. Then $-f^{-1}(X) = f^{-1}(-X)$.

(3) For every 1-sorted structure T and for every family F of subsets of T holds $F^{\mathrm{c}} = \{-a; a$ ranges over subsets of T: $a \in F\}$.

(4) Let R be a non empty relational structure and F be a subset of R. Then $\uparrow F = \bigcup\{\uparrow x; x$ ranges over elements of R: $x \in F\}$ and $\downarrow F = \bigcup\{\downarrow x; x$ ranges over elements of R: $x \in F\}$.

(5) Let R be a non empty relational structure, A be a family of subsets of R, and F be a subset of R. If $A = \{-\uparrow x; x$ ranges over elements of R: $x \in F\}$, then $\mathrm{Intersect}(A) = -\uparrow F$.

Let us mention that there exists a FR-structure which is strict, trivial, reflexive, non empty, discrete, and finite.

One can check that there exists a top-lattice which is strict, complete, and trivial.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is sups-preserving.

Let R, S be 1-sorted structures. Let us assume that the carrier of $S \subseteq$ the carrier of R. The functor $\mathrm{incl}(S, R)$ yields a map from S into R and is defined as follows:

(Def. 1) $\mathrm{incl}(S, R) = \mathrm{id}_{\text{the carrier of } S}$.

Let R be a non empty relational structure and let S be a non empty relational substructure of R. One can check that $\mathrm{incl}(S, R)$ is monotone.

Let R be a non empty relational structure and let X be a non empty subset of the carrier of R. Note that $\mathrm{sub}(X)$ is non empty.

Let R be a non empty relational structure and let X be a non empty subset of the carrier of R. The functor $\langle X; \mathrm{id}\rangle$ yielding a strict non empty net structure over R is defined as follows:

(Def. 2) $\langle X; \mathrm{id}\rangle = \mathrm{incl}(\mathrm{sub}(X), R) \cdot \langle \mathrm{sub}(X); \mathrm{id}\rangle$.

The functor $\langle X^{\mathrm{op}}; \mathrm{id}\rangle$ yielding a strict non empty net structure over R is defined as follows:

(Def. 3) $\langle X^{\rm op}; {\rm id}\rangle = {\rm incl}({\rm sub}(X), R) \cdot \langle ({\rm sub}(X))^{\rm op}; {\rm id}\rangle$.

One can prove the following propositions:

(6) Let R be a non empty relational structure and X be a non empty subset of R. Then

(i) the carrier of $\langle X; {\rm id}\rangle = X$,

(ii) $\langle X; {\rm id}\rangle$ is a full relational substructure of R, and

(iii) for every element x of $\langle X; {\rm id}\rangle$ holds $\langle X; {\rm id}\rangle(x) = x$.

(7) Let R be a non empty relational structure and X be a non empty subset of R. Then

(i) the carrier of $\langle X^{\rm op}; {\rm id}\rangle = X$,

(ii) $\langle X^{\rm op}; {\rm id}\rangle$ is a full relational substructure of $R^{\rm op}$, and

(iii) for every element x of $\langle X^{\rm op}; {\rm id}\rangle$ holds $\langle X^{\rm op}; {\rm id}\rangle(x) = x$.

Let R be a non empty reflexive relational structure and let X be a non empty subset of R. One can check that $\langle X; {\rm id}\rangle$ is reflexive and $\langle X^{\rm op}; {\rm id}\rangle$ is reflexive.

Let R be a non empty transitive relational structure and let X be a non empty subset of R. Observe that $\langle X; {\rm id}\rangle$ is transitive and $\langle X^{\rm op}; {\rm id}\rangle$ is transitive.

Let R be a non empty reflexive relational structure and let I be a directed subset of R. Note that ${\rm sub}(I)$ is directed.

Let R be a non empty reflexive relational structure and let I be a directed non empty subset of R. Note that $\langle I; {\rm id}\rangle$ is directed.

Let R be a non empty reflexive relational structure and let F be a filtered non empty subset of R. One can verify that $\langle ({\rm sub}(F))^{\rm op}; {\rm id}\rangle$ is directed.

Let R be a non empty reflexive relational structure and let F be a filtered non empty subset of R. Note that $\langle F^{\rm op}; {\rm id}\rangle$ is directed.

2. Operations on Families of Open Sets

One can prove the following propositions:

(8) For every topological space T such that T is empty holds the topology of $T = \{\emptyset\}$.

(9) Let T be a trivial non empty topological space. Then

(i) the topology of $T = 2^{\text{the carrier of } T}$, and

(ii) for every point x of T holds the carrier of $T = \{x\}$ and the topology of $T = \{\emptyset, \{x\}\}$.

(10) Let T be a trivial non empty topological space. Then $\{$the carrier of $T\}$ is a basis of T and $\emptyset$ is a prebasis of T and $\{\emptyset\}$ is a prebasis of T.

(11) For all sets X, Y and for every family A of subsets of X such that $A = \{Y\}$ holds ${\rm FinMeetCl}(A) = \{Y, X\}$ and ${\rm UniCl}(A) = \{Y, \emptyset\}$.

(12) For every set X and for all families A, B of subsets of X such that $A = B \cup \{X\}$ or $B = A \setminus \{X\}$ holds $\mathrm{Intersect}(A) = \mathrm{Intersect}(B)$.

(13) For every set X and for all families A, B of subsets of X such that $A = B \cup \{X\}$ or $B = A \setminus \{X\}$ holds $\mathrm{FinMeetCl}(A) = \mathrm{FinMeetCl}(B)$.

(14) Let X be a set and A be a family of subsets of X. Suppose $X \in A$. Let x be a set. Then $x \in \mathrm{FinMeetCl}(A)$ if and only if there exists a finite non empty family Y of subsets of X such that $Y \subseteq A$ and $x = \mathrm{Intersect}(Y)$.

(15) For every set X and for every family A of subsets of X holds $\mathrm{UniCl}(\mathrm{UniCl}(A)) = \mathrm{UniCl}(A)$.

(16) For every set X and for every empty family A of subsets of X holds $\mathrm{UniCl}(A) = \{\emptyset\}$.

(17) For every set X and for every empty family A of subsets of X holds $\mathrm{FinMeetCl}(A) = \{X\}$.

(18) For every set X and for every family A of subsets of X such that $A = \{\emptyset, X\}$ holds $\mathrm{UniCl}(A) = A$ and $\mathrm{FinMeetCl}(A) = A$.

(19) Let X, Y be sets, A be a family of subsets of X, and B be a family of subsets of Y. Then
   (i) if $A \subseteq B$, then $\mathrm{UniCl}(A) \subseteq \mathrm{UniCl}(B)$, and
   (ii) if $A = B$, then $\mathrm{UniCl}(A) = \mathrm{UniCl}(B)$.

(20) Let X, Y be sets, A be a family of subsets of X, and B be a family of subsets of Y. If $A = B$ and $X \in A$ and $X \subseteq Y$, then $\mathrm{FinMeetCl}(B) = \{Y\} \cup \mathrm{FinMeetCl}(A)$.

(21) For every set X and for every family A of subsets of X holds $\mathrm{UniCl}(\mathrm{FinMeetCl}(\mathrm{UniCl}(A))) = \mathrm{UniCl}(\mathrm{FinMeetCl}(A))$.

3. Bases

Next we state a number of propositions:

(22) Let T be a topological space and K be a family of subsets of T. Then the topology of $T = \mathrm{UniCl}(K)$ if and only if K is a basis of T.

(23) Let T be a topological space and K be a family of subsets of the carrier of T. Then K is a prebasis of T if and only if $\mathrm{FinMeetCl}(K)$ is a basis of T.

(24) Let T be a non empty topological space and B be a family of subsets of T. If $\mathrm{UniCl}(B)$ is a prebasis of T, then B is a prebasis of T.

(25) Let T_1, T_2 be topological spaces and B be a basis of T_1. Suppose the carrier of $T_1 =$ the carrier of T_2 and B is a basis of T_2. Then the topology of $T_1 =$ the topology of T_2.

(26) Let T_1, T_2 be topological spaces and P be a prebasis of T_1. Suppose the carrier of T_1 = the carrier of T_2 and P is a prebasis of T_2. Then the topology of T_1 = the topology of T_2.

(27) For every topological space T holds every basis of T is open and is a prebasis of T.

(28) For every topological space T holds every prebasis of T is open.

(29) Let T be a non empty topological space and B be a prebasis of T. Then $B \cup \{$the carrier of $T\}$ is a prebasis of T.

(30) For every topological space T and for every basis B of T holds $B \cup \{$the carrier of $T\}$ is a basis of T.

(31) Let T be a topological space, B be a basis of T, and A be a subset of T. Then A is open if and only if for every point p of T such that $p \in A$ there exists a subset a of T such that $a \in B$ and $p \in a$ and $a \subseteq A$.

(32) Let T be a topological space and B be a family of subsets of T. Suppose that

(i) $B \subseteq$ the topology of T, and

(ii) for every subset A of T such that A is open and for every point p of T such that $p \in A$ there exists a subset a of T such that $a \in B$ and $p \in a$ and $a \subseteq A$.

Then B is a basis of T.

(33) Let S be a topological space, T be a non empty topological space, K be a basis of T, and f be a map from S into T. Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(-A)$ is closed.

(34) Let S be a topological space, T be a non empty topological space, K be a basis of T, and f be a map from S into T. Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A)$ is open.

(35) Let S be a topological space, T be a non empty topological space, K be a prebasis of T, and f be a map from S into T. Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(-A)$ is closed.

(36) Let S be a topological space, T be a non empty topological space, K be a prebasis of T, and f be a map from S into T. Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A)$ is open.

(37) Let T be a non empty topological space, x be a point of T, X be a subset of T, and K be a basis of T. Suppose that for every subset A of T such that $A \in K$ and $x \in A$ holds A meets X. Then $x \in \overline{X}$.

(38) Let T be a non empty topological space, x be a point of T, X be a subset of T, and K be a prebasis of T. Suppose that for every finite family Z of subsets of T such that $Z \subseteq K$ and $x \in \mathrm{Intersect}(Z)$ holds $\mathrm{Intersect}(Z)$ meets X. Then $x \in \overline{X}$.

(39) Let T be a non empty topological space, K be a prebasis of T, x be a point of T, and N be a net in T. Suppose that for every subset A of T such that $A \in K$ and $x \in A$ holds N is eventually in A. Let S be a subset of T. If $\mathrm{rng}\,\mathrm{netmap}(N,T) \subseteq S$, then $x \in \overline{S}$.

4. Product Topologies

The following four propositions are true:

(40) Let T_1, T_2 be non empty topological spaces, B_1 be a basis of T_1, and B_2 be a basis of T_2. Then $\{[\!: a, b :\!]; a$ ranges over subsets of T_1, b ranges over subsets of T_2: $a \in B_1 \ \wedge \ b \in B_2\}$ is a basis of $[\!: T_1, T_2 :\!]$.

(41) Let T_1, T_2 be non empty topological spaces, B_1 be a prebasis of T_1, and B_2 be a prebasis of T_2. Then $\{[\!:$ the carrier of $T_1,\ b :\!]; b$ ranges over subsets of T_2: $b \in B_2\} \cup \{[\!: a,$ the carrier of $T_2 :\!]; a$ ranges over subsets of T_1: $a \in B_1\}$ is a prebasis of $[\!: T_1, T_2 :\!]$.

(42) Let X_1, X_2 be sets, A be a family of subsets of $[\!: X_1, X_2 :\!]$, A_1 be a non empty family of subsets of X_1, and A_2 be a non empty family of subsets of X_2. Suppose $A = \{[\!: a, b :\!]; a$ ranges over subsets of X_1, b ranges over subsets of X_2: $a \in A_1 \ \wedge \ b \in A_2\}$. Then $\mathrm{Intersect}(A) = [\!: \mathrm{Intersect}(A_1), \mathrm{Intersect}(A_2) :\!]$.

(43) Let T_1, T_2 be non empty topological spaces, B_1 be a prebasis of T_1, and B_2 be a prebasis of T_2. Suppose $\bigcup B_1 =$ the carrier of T_1 and $\bigcup B_2 =$ the carrier of T_2. Then $\{[\!: a, b :\!]; a$ ranges over subsets of T_1, b ranges over subsets of T_2: $a \in B_1 \ \wedge \ b \in B_2\}$ is a prebasis of $[\!: T_1, T_2 :\!]$.

5. Topological Augmentations

Let R be a relational structure. A FR-structure is called a topological augmentation of R if:

(Def. 4) The relational structure of it = the relational structure of R.

Let R be a relational structure and let T be a topological augmentation of R. We introduce T is correct as a synonym of T is topological space-like.

Let R be a relational structure. Note that there exists a topological augmentation of R which is correct, discrete, and strict.

We now state three propositions:

(44) Every FR-structure T is a topological augmentation of T.

(45) Let S be a FR-structure and T be a topological augmentation of S. Then S is a topological augmentation of T.

(46) Let R be a relational structure and T_1 be a topological augmentation of R. Then every topological augmentation of T_1 is a topological augmentation of R.

Let R be a non empty relational structure. One can check that every topological augmentation of R is non empty.

Let R be a reflexive relational structure. Note that every topological augmentation of R is reflexive.

Let R be a transitive relational structure. One can check that every topological augmentation of R is transitive.

Let R be an antisymmetric relational structure. One can verify that every topological augmentation of R is antisymmetric.

Let R be a complete non empty relational structure. Observe that every topological augmentation of R is complete.

We now state three propositions:

(47) Let S be a complete reflexive antisymmetric non empty relational structure and T be a non empty reflexive relational structure. Suppose the relational structure of $S =$ the relational structure of T. Let A be a subset of S and C be a subset of T. If $A = C$ and A is inaccessible, then C is inaccessible.

(48) Let S be a non empty reflexive relational structure and T be a topological augmentation of S. If the topology of $T = \sigma(S)$, then T is correct.

(49) Let S be a complete lattice and T be a topological augmentation of S. If the topology of $T = \sigma(S)$, then T is Scott.

Let R be a complete lattice. One can verify that there exists a topological augmentation of R which is Scott, strict, and correct.

The following three propositions are true:

(50) Let S, T be complete Scott non empty reflexive transitive antisymmetric FR-structures. Suppose the relational structure of $S =$ the relational structure of T. Let F be a subset of S and G be a subset of T. If $F = G$, then if F is open, then G is open.

(51) For every complete lattice S and for every Scott topological augmentation T of S holds the topology of $T = \sigma(S)$.

(52) Let S, T be complete lattices. Suppose the relational structure of $S =$ the relational structure of T. Then $\sigma(S) = \sigma(T)$.

Let R be a complete lattice. Observe that every topological augmentation of R which is Scott is also correct.

6. Refinements

Let T be a topological structure. A topological space is said to be a topological extension of T if:

(Def. 5) The carrier of T = the carrier of it and the topology of $T \subseteq$ the topology of it.

One can prove the following proposition

(53) Let S be a topological structure. Then there exists a topological extension T of S such that T is strict and the topology of S is a prebasis of T.

Let T be a topological structure. Note that there exists a topological extension of T which is strict and discrete.

Let T_1, T_2 be topological structures. A topological space is said to be a refinement of T_1 and T_2 if it satisfies the conditions (Def. 6).

(Def. 6)(i) The carrier of it = (the carrier of T_1) $\cup$ (the carrier of T_2), and

(ii) (the topology of T_1) $\cup$ (the topology of T_2) is a prebasis of it.

Let T_1 be a non empty topological structure and let T_2 be a topological structure. Observe that every refinement of T_1 and T_2 is non empty and every refinement of T_2 and T_1 is non empty.

The following propositions are true:

(54) Let T_1, T_2 be topological structures and T, T' be refinements of T_1 and T_2. Then the topological structure of T = the topological structure of T'.

(55) For all topological structures T_1, T_2 holds every refinement of T_1 and T_2 is a refinement of T_2 and T_1.

(56) Let T_1, T_2 be topological structures, T be a refinement of T_1 and T_2, and X be a family of subsets of T. Suppose X = (the topology of T_1) $\cup$ (the topology of T_2). Then the topology of T = UniCl(FinMeetCl(X)).

(57) Let T_1, T_2 be topological structures. Suppose the carrier of T_1 = the carrier of T_2. Then every refinement of T_1 and T_2 is a topological extension of T_1 and a topological extension of T_2.

(58) Let T_1, T_2 be non empty topological spaces, T be a refinement of T_1 and T_2, B_1 be a prebasis of T_1, and B_2 be a prebasis of T_2. Then $B_1 \cup B_2 \cup \{$the carrier of T_1, the carrier of $T_2\}$ is a prebasis of T.

(59) Let T_1, T_2 be non empty topological spaces, B_1 be a basis of T_1, B_2 be a basis of T_2, and T be a refinement of T_1 and T_2. Then $B_1 \cup B_2 \cup B_1 \Cap B_2$ is a basis of T.

(60) Let T_1, T_2 be non empty topological spaces. Suppose the carrier of T_1 = the carrier of T_2. Let T be a refinement of T_1 and T_2. Then (the topology of T_1) $\Cap$ (the topology of T_2) is a basis of T.

(61) Let L be a non empty relational structure, T_1, T_2 be correct topological augmentations of L, and T be a refinement of T_1 and T_2. Then (the topology of T_1) $\Cap$ (the topology of T_2) is a basis of T.

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