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A Theory of Boolean Valued Functions and Partitions

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Summary. In this paper, we define Boolean valued functions. Some of their algebraic properties are proved. We also introduce and examine the infimum and supremum of Boolean valued functions and their properties. In the last section, relations between Boolean valued functions and partitions are discussed.

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The terminology and notation used in this paper are introduced in the following papers: [4], [6], [1], [2], [3], and [5].

1. Boolean Operations

In this paper Y denotes a non empty set.

Let k, l be elements of *Boolean*. The functor $k \Rightarrow l$ is defined by:

(Def. 1) $k \Rightarrow l = \neg k \vee l$.

The functor $k \Leftrightarrow l$ is defined as follows:

(Def. 2) $k \Leftrightarrow l = \neg(k \oplus l)$.

Let k, l be elements of *Boolean*. The predicate $k \Subset l$ is defined by:

(Def. 3) $k \Rightarrow l = true$.

Let us note that the predicate $k \Subset l$ is reflexive.

One can prove the following three propositions:

(1) For all elements k, l of *Boolean* and for all natural numbers n_1, n_2 such that $k = n_1$ and $l = n_2$ holds $k \Subset l$ iff $n_1 \leqslant n_2$.

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(2) For all elements k, l of *Boolean* such that $k \Subset l$ and $l \Subset k$ holds $k = l$.

(3) For all elements k, l, m of *Boolean* such that $k \Subset l$ and $l \Subset m$ holds $k \Subset m$.

2. Boolean Valued Functions

Let us consider Y. The functor $\mathrm{BVF}(Y)$ is defined by:

(Def. 4) $\mathrm{BVF}(Y) = Boolean^Y$.

Let Y be a non empty set. Observe that $\mathrm{BVF}(Y)$ is functional and non empty.

Let us consider Y, let a be an element of $\mathrm{BVF}(Y)$, and let x be an element of Y. The functor $\mathrm{Pj}(a, x)$ yields an element of *Boolean* and is defined by:

(Def. 5) $\mathrm{Pj}(a, x) = a(x)$.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. The functor $a \wedge b$ yields an element of $\mathrm{BVF}(Y)$ and is defined by:

(Def. 6) For every element x of Y holds $\mathrm{Pj}(a \wedge b, x) = \mathrm{Pj}(a, x) \wedge \mathrm{Pj}(b, x)$.

Let us notice that the functor $a \wedge b$ is commutative.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. The functor $a \vee b$ yields an element of $\mathrm{BVF}(Y)$ and is defined by:

(Def. 7) For every element x of Y holds $\mathrm{Pj}(a \vee b, x) = \mathrm{Pj}(a, x) \vee \mathrm{Pj}(b, x)$.

Let us notice that the functor $a \vee b$ is commutative.

Let us consider Y and let a be an element of $\mathrm{BVF}(Y)$. The functor $\neg a$ yielding an element of $\mathrm{BVF}(Y)$ is defined as follows:

(Def. 8) For every element x of Y holds $\mathrm{Pj}(\neg a, x) = \neg\,\mathrm{Pj}(a, x)$.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. The functor $a \oplus b$ yields an element of $\mathrm{BVF}(Y)$ and is defined as follows:

(Def. 9) For every element x of Y holds $\mathrm{Pj}(a \oplus b, x) = \mathrm{Pj}(a, x) \oplus \mathrm{Pj}(b, x)$.

Let us note that the functor $a \oplus b$ is commutative.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. The functor $a \Rightarrow b$ yields an element of $\mathrm{BVF}(Y)$ and is defined by:

(Def. 10) For every element x of Y holds $\mathrm{Pj}(a \Rightarrow b, x) = \neg\,\mathrm{Pj}(a, x) \vee \mathrm{Pj}(b, x)$.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. The functor $a \Leftrightarrow b$ yielding an element of $\mathrm{BVF}(Y)$ is defined as follows:

(Def. 11) For every element x of Y holds $\mathrm{Pj}(a \Leftrightarrow b, x) = \neg(\mathrm{Pj}(a, x) \oplus \mathrm{Pj}(b, x))$.

Let us observe that the functor $a \Leftrightarrow b$ is commutative.

Let us consider Y. The functor $\mathit{false}(Y)$ yielding an element of $\mathrm{BVF}(Y)$ is defined by:

(Def. 12) For every element x of Y holds $\mathrm{Pj}(\mathit{false}(Y), x) = \mathit{false}$.

Let us consider Y. The functor $true(Y)$ yielding an element of BVF(Y) is defined as follows:

(Def. 13) For every element x of Y holds $\mathrm{Pj}(true(Y), x) = true$.

The following propositions are true:

(4) For every element a of BVF(Y) holds $\neg\neg a = a$.

(5) For every element a of BVF(Y) holds $\neg\, true(Y) = false(Y)$ and $\neg\, false(Y) = true(Y)$.

(6) For all elements a, b of BVF(Y) holds $a \wedge a = a$.

(7) For all elements a, b, c of BVF(Y) holds $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

(8) For every element a of BVF(Y) holds $a \wedge false(Y) = false(Y)$.

(9) For every element a of BVF(Y) holds $a \wedge true(Y) = a$.

(10) For every element a of BVF(Y) holds $a \vee a = a$.

(11) For all elements a, b, c of BVF(Y) holds $(a \vee b) \vee c = a \vee (b \vee c)$.

(12) For every element a of BVF(Y) holds $a \vee false(Y) = a$.

(13) For every element a of BVF(Y) holds $a \vee true(Y) = true(Y)$.

(14) For all elements a, b, c of BVF(Y) holds $a \wedge b \vee c = (a \vee c) \wedge (b \vee c)$.

(15) For all elements a, b, c of BVF(Y) holds $(a \vee b) \wedge c = a \wedge c \vee b \wedge c$.

(16) For all elements a, b of BVF(Y) holds $\neg(a \vee b) = \neg a \wedge \neg b$.

(17) For all elements a, b of BVF(Y) holds $\neg(a \wedge b) = \neg a \vee \neg b$.

Let us consider Y and let a, b be elements of BVF(Y). The predicate $a \Subset b$ is defined by:

(Def. 14) For every element x of Y such that $\mathrm{Pj}(a, x) = true$ holds $\mathrm{Pj}(b, x) = true$.

Let us note that the predicate $a \Subset b$ is reflexive.

The following four propositions are true:

(18) For all elements a, b, c of BVF(Y) holds if $a \Subset b$ and $b \Subset a$, then $a = b$ and if $a \Subset b$ and $b \Subset c$, then $a \Subset c$.

(19) For all elements a, b of BVF(Y) holds $a \Rightarrow b = true(Y)$ iff $a \Subset b$.

(20) For all elements a, b of BVF(Y) holds $a \Leftrightarrow b = true(Y)$ iff $a = b$.

(21) For every element a of BVF(Y) holds $false(Y) \Subset a$ and $a \Subset true(Y)$.

3. Infimum and Supremum

Let us consider Y and let a be an element of BVF(Y). The functor $\mathrm{INF}\, a$ yields an element of BVF(Y) and is defined as follows:

(Def. 15) $\mathrm{INF}\, a = \begin{cases} true(Y), & \text{if for every element } x \text{ of } Y \text{ holds } \mathrm{Pj}(a, x) = true, \\ false(Y), & \text{otherwise.} \end{cases}$

The functor $\mathrm{SUP}\, a$ yielding an element of BVF(Y) is defined by:

(Def. 16) $\mathrm{SUP}\,a = \begin{cases} \mathit{false}(Y), & \text{if for every element } x \text{ of } Y \text{ holds } \mathrm{Pj}(a,x) = \mathit{false}, \\ \mathit{true}(Y), & \text{otherwise.} \end{cases}$

Next we state two propositions:

(22) For every element a of $\mathrm{BVF}(Y)$ holds $\neg\,\mathrm{INF}\,a = \mathrm{SUP}\,\neg a$ and $\neg\,\mathrm{SUP}\,a = \mathrm{INF}\,\neg a$.

(23) $\mathrm{INF}\,\mathit{false}(Y) = \mathit{false}(Y)$ and $\mathrm{INF}\,\mathit{true}(Y) = \mathit{true}(Y)$ and $\mathrm{SUP}\,\mathit{false}(Y) = \mathit{false}(Y)$ and $\mathrm{SUP}\,\mathit{true}(Y) = \mathit{true}(Y)$.

Let us consider Y. Observe that $\mathit{false}(Y)$ is constant.

Let us consider Y. One can verify that $\mathit{true}(Y)$ is constant.

Let Y be a non empty set. Observe that there exists an element of $\mathrm{BVF}(Y)$ which is constant.

We now state several propositions:

(24) For every constant element a of $\mathrm{BVF}(Y)$ holds $a = \mathit{false}(Y)$ or $a = \mathit{true}(Y)$.

(25) For every constant element d of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}\,d = d$ and $\mathrm{SUP}\,d = d$.

(26) For all elements a, b of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}(a \wedge b) = \mathrm{INF}\,a \wedge \mathrm{INF}\,b$ and $\mathrm{SUP}(a \vee b) = \mathrm{SUP}\,a \vee \mathrm{SUP}\,b$.

(27) For every element a of $\mathrm{BVF}(Y)$ and for every constant element d of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}(d \Rightarrow a) = d \Rightarrow \mathrm{INF}\,a$ and $\mathrm{INF}(a \Rightarrow d) = \mathrm{SUP}\,a \Rightarrow d$.

(28) For every element a of $\mathrm{BVF}(Y)$ and for every constant element d of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}(d \vee a) = d \vee \mathrm{INF}\,a$ and $\mathrm{SUP}(d \wedge a) = d \wedge \mathrm{SUP}\,a$ and $\mathrm{SUP}(a \wedge d) = \mathrm{SUP}\,a \wedge d$.

(29) For every element a of $\mathrm{BVF}(Y)$ and for every element x of Y holds $\mathrm{Pj}(\mathrm{INF}\,a, x) \Subset \mathrm{Pj}(a, x)$.

(30) For every element a of $\mathrm{BVF}(Y)$ and for every element x of Y holds $\mathrm{Pj}(a, x) \Subset \mathrm{Pj}(\mathrm{SUP}\,a, x)$.

4. Boolean Valued Functions and Partitions

Let us consider Y, let a be an element of $\mathrm{BVF}(Y)$, and let P_1 be a partition of Y. We say that a is dependent of P_1 if and only if:

(Def. 17) For every set F such that $F \in P_1$ and for all sets x_1, x_2 such that $x_1 \in F$ and $x_2 \in F$ holds $a(x_1) = a(x_2)$.

The following two propositions are true:

(31) For every element a of $\mathrm{BVF}(Y)$ holds a is dependent of $\mathcal{I}(Y)$.

(32) For every constant element a of $\mathrm{BVF}(Y)$ holds a is dependent of $\mathcal{O}(Y)$.

Let us consider Y and let P_1 be a partition of Y. We see that the element of P_1 is a subset of Y.

Let us consider Y, let x be an element of Y, and let P_1 be a partition of Y. Then $\mathrm{EqClass}(x, P_1)$ is an element of P_1. We introduce $\mathrm{Lift}(x, P_1)$ as a synonym of $\mathrm{EqClass}(x, P_1)$.

Let us consider Y, let a be an element of $\mathrm{BVF}(Y)$, and let P_1 be a partition of Y. The functor $\mathrm{INF}(a, P_1)$ yields an element of $\mathrm{BVF}(Y)$ and is defined by the condition (Def. 18).

(Def. 18) Let y be an element of Y. Then

(i) if for every element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ holds $\mathrm{Pj}(a, x) = \mathit{true}$, then $\mathrm{Pj}(\mathrm{INF}(a, P_1), y) = \mathit{true}$, and

(ii) if it is not true that for every element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ holds $\mathrm{Pj}(a, x) = \mathit{true}$, then $\mathrm{Pj}(\mathrm{INF}(a, P_1), y) = \mathit{false}$.

Let us consider Y, let a be an element of $\mathrm{BVF}(Y)$, and let P_1 be a partition of Y. The functor $\mathrm{SUP}(a, P_1)$ yielding an element of $\mathrm{BVF}(Y)$ is defined by the condition (Def. 19).

(Def. 19) Let y be an element of Y. Then

(i) if there exists an element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ and $\mathrm{Pj}(a, x) = \mathit{true}$, then $\mathrm{Pj}(\mathrm{SUP}(a, P_1), y) = \mathit{true}$, and

(ii) if it is not true that there exists an element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ and $\mathrm{Pj}(a, x) = \mathit{true}$, then $\mathrm{Pj}(\mathrm{SUP}(a, P_1), y) = \mathit{false}$.

Next we state a number of propositions:

(33) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\mathrm{INF}(a, P_1)$ is dependent of P_1.

(34) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\mathrm{SUP}(a, P_1)$ is dependent of P_1.

(35) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\mathrm{INF}(a, P_1) \Subset a$.

(36) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $a \Subset \mathrm{SUP}(a, P_1)$.

(37) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\neg\,\mathrm{INF}(a, P_1) = \mathrm{SUP}(\neg a, P_1)$.

(38) For every element a of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}(a, \mathcal{O}(Y)) = \mathrm{INF}\, a$.

(39) For every element a of $\mathrm{BVF}(Y)$ holds $\mathrm{SUP}(a, \mathcal{O}(Y)) = \mathrm{SUP}\, a$.

(40) For every element a of $\mathrm{BVF}(Y)$ holds $\mathrm{INF}(a, \mathcal{I}(Y)) = a$.

(41) For every element a of $\mathrm{BVF}(Y)$ holds $\mathrm{SUP}(a, \mathcal{I}(Y)) = a$.

(42) For all elements a, b of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\mathrm{INF}(a \wedge b, P_1) = \mathrm{INF}(a, P_1) \wedge \mathrm{INF}(b, P_1)$.

(43) For all elements a, b of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y holds $\mathrm{SUP}(a \vee b, P_1) = \mathrm{SUP}(a, P_1) \vee \mathrm{SUP}(b, P_1)$.

Let us consider Y and let f be an element of $\mathrm{BVF}(Y)$. The functor $\mathrm{GPart}\, f$ yields a partition of Y and is defined by:

(Def. 20) $\mathrm{GPart}\, f = \{\{x; x$ ranges over elements of Y: $f(x) = \mathit{true}\}, \{x'; x'$ ranges over elements of Y: $f(x') = \mathit{false}\}\} \setminus \{\emptyset\}$.

The following propositions are true:

(44) For every element a of $\mathrm{BVF}(Y)$ holds a is dependent of $\mathrm{GPart}\, a$.

(45) For every element a of $\mathrm{BVF}(Y)$ and for every partition P_1 of Y such that a is dependent of P_1 holds P_1 is finer than $\mathrm{GPart}\, a$.

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