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Trigonometric Functions and Existence of Circle Ratio

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Summary. In this article, we defined *sinus* and *cosine* as the real part and the imaginary part of the exponential function on complex, and also give their series expression. Then we proved the differentiablity of *sinus*, *cosine* and the exponential function of real. Finally, we showed the existence of the circle ratio, and some formulas of *sinus*, *cosine*.

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The papers [11], [3], [1], [10], [17], [14], [15], [4], [5], [2], [12], [16], [6], [20], [21], [8], [9], [7], [13], [18], and [19] provide the terminology and notation for this paper.

1. Some Definitions and Properties of Complex Sequence

For simplicity, we adopt the following rules: p, q, r, t_1, t_2, t_3 are elements of $\mathbb{R}$, w, z, z_1, z_2 are elements of $\mathbb{C}$, k, l, m, n are natural numbers, s_1 is a complex sequence, and r_1 is a sequence of real numbers.

Let m, k be natural numbers. Let us assume that $k \leqslant m$. The functor $\mathrm{PN}(m,k)$ yielding an element of $\mathbb{N}$ is defined by:

(Def. 1) $\mathrm{PN}(m,k) = m - k$.

Let m, k be natural numbers. The functor $\mathrm{CHK}(m,k)$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 2) $\mathrm{CHK}(m,k) = \begin{cases} 1_{\mathbb{C}}, & \text{if } m \leqslant k, \\ 0_{\mathbb{C}}, & \text{otherwise.} \end{cases}$

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The functor RHK(m, k) yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 3) $\text{RHK}(m, k) = \begin{cases} 1, & \text{if } m \leqslant k, \\ 0, & \text{otherwise.} \end{cases}$

In this article we present several logical schemes. The scheme *ExComplex CASE* deals with a binary functor $\mathcal{F}$ yielding an element of $\mathbb{C}$, and states that:

For every k there exists s_1 such that for every n holds if $n \leqslant k$, then $s_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $s_1(n) = 0_{\mathbb{C}}$

for all values of the parameter.

The scheme *ExReal CASE* deals with a binary functor $\mathcal{F}$ yielding an element of $\mathbb{R}$, and states that:

For every k there exists r_1 such that for every n holds if $n \leqslant k$, then $r_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $r_1(n) = 0$

for all values of the parameter.

The complex sequence Prod_complex_n is defined by:

(Def. 4) $(\text{Prod\_complex\_n})(0) = 1_{\mathbb{C}}$ and for every n holds $(\text{Prod\_complex\_n})(n + 1) = (\text{Prod\_complex\_n})(n) \cdot ((n + 1) + 0i)$.

The sequence Prod_real_n of real numbers is defined by:

(Def. 5) $(\text{Prod\_real\_n})(0) = 1$ and for every n holds $(\text{Prod\_real\_n})(n + 1) = (\text{Prod\_real\_n})(n) \cdot (n + 1)$.

Let n be a natural number. The functor $n!c$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 6) $n!c = (\text{Prod\_complex\_n})(n)$.

Let n be a natural number. Then $n!$ is a real number and it can be characterized by the condition:

(Def. 7) $n! = (\text{Prod\_real\_n})(n)$.

Let z be an element of $\mathbb{C}$. The functor $z\,\text{ExpSeq}$ yields a complex sequence and is defined as follows:

(Def. 8) For every n holds $z\,\text{ExpSeq}(n) = \frac{z_{\mathbb{N}}^{n}}{n!c}$.

Let a be an element of $\mathbb{R}$. The functor $a\,\text{ExpSeq}$ yielding a sequence of real numbers is defined as follows:

(Def. 9) For every n holds $a\,\text{ExpSeq}(n) = \frac{a_{\mathbb{N}}^{n}}{n!}$.

The following propositions are true:

(1) If $0 < n$, then $n + 0i \neq 0_{\mathbb{C}}$ and $0!c = 1_{\mathbb{C}}$ and $n!c \neq 0_{\mathbb{C}}$ and $n + 1!c = n!c \cdot ((n + 1) + 0i)$.

(2) $n! \neq 0$ and $(n + 1)! = n! \cdot (n + 1)$.

(3) For every k such that $0 < k$ holds $\text{PN}(k, 1)!c \cdot (k + 0i) = k!c$ and for all m, k such that $k \leqslant m$ holds $\text{PN}(m, k)!c \cdot (((m + 1) - k) + 0i) = \text{PN}(m + 1, k)!c$.

Let n be a natural number. The functor Coef n yielding a complex sequence is defined by:

(Def. 10) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef}\, n)(k) = \frac{n!c}{k!c \cdot \mathrm{PN}(n,k)!c}$ and if $k > n$, then $(\mathrm{Coef}\, n)(k) = 0_{\mathbb{C}}$.

Let n be a natural number. The functor Coef_e n yields a complex sequence and is defined as follows:

(Def. 11) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef\_e}\, n)(k) = \frac{1_{\mathbb{C}}}{k!c \cdot \mathrm{PN}(n,k)!c}$ and if $k > n$, then $(\mathrm{Coef\_e}\, n)(k) = 0_{\mathbb{C}}$.

Let us consider s_1. The functor Sift s_1 yielding a complex sequence is defined as follows:

(Def. 12) $(\mathrm{Sift}\, s_1)(0) = 0_{\mathbb{C}}$ and for every natural number k holds $(\mathrm{Sift}\, s_1)(k+1) = s_1(k)$.

Let us consider n and let z, w be elements of $\mathbb{C}$. The functor $\mathrm{Expan}(n, z, w)$ yields a complex sequence and is defined as follows:

(Def. 13) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan}(n, z, w))(k) = (\mathrm{Coef}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{\mathrm{PN}(n,k)}$ and if $n < k$, then $(\mathrm{Expan}(n, z, w))(k) = 0_{\mathbb{C}}$.

Let us consider n and let z, w be elements of $\mathbb{C}$. The functor $\mathrm{Expan\_e}(n, z, w)$ yielding a complex sequence is defined by:

(Def. 14) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan\_e}(n, z, w))(k) = (\mathrm{Coef\_e}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{\mathrm{PN}(n,k)}$ and if $n < k$, then $(\mathrm{Expan\_e}(n, z, w))(k) = 0_{\mathbb{C}}$.

Let us consider n and let z, w be elements of $\mathbb{C}$. The functor $\mathrm{Alfa}(n, z, w)$ yielding a complex sequence is defined by:

(Def. 15) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Alfa}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(\mathrm{PN}(n, k))$ and if $n < k$, then $(\mathrm{Alfa}(n, z, w))(k) = 0_{\mathbb{C}}$.

Let a, b be elements of $\mathbb{R}$ and let n be a natural number. The functor $\mathrm{Conj}(n, a, b)$ yielding a sequence of real numbers is defined as follows:

(Def. 16) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n, a, b))(k) = a\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} b\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} b\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(\mathrm{PN}(n, k)))$ and if $n < k$, then $(\mathrm{Conj}(n, a, b))(k) = 0$.

Let z, w be elements of $\mathbb{C}$ and let n be a natural number. The functor $\mathrm{Conj}(n, z, w)$ yielding a complex sequence is defined by:

(Def. 17) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(\mathrm{PN}(n, k)))$ and if $n < k$, then $(\mathrm{Conj}(n, z, w))(k) = 0_{\mathbb{C}}$.

The following propositions are true:

(4) $z\,\mathrm{ExpSeq}(n+1) = \frac{z\,\mathrm{ExpSeq}(n) \cdot z}{(n+1)+0i}$ and $z\,\mathrm{ExpSeq}(0) = 1_{\mathbb{C}}$ and $|z\,\mathrm{ExpSeq}(n)| = |z|\,\mathrm{ExpSeq}(n)$.

(5) If $0 < k$, then $(\mathrm{Sift}\, s_1)(k) = s_1(\mathrm{PN}(k, 1))$.

(6) $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Sift}\, s_1)(\alpha))_{\kappa \in \mathbb{N}}(k) + s_1(k)$.

(7) $(z + w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(8) $\mathrm{Expan\_e}(n, z, w) = \frac{1_{\mathbb{C}}}{n!c} \mathrm{Expan}(n, z, w)$.

(9) $\frac{(z+w)^n_{\mathbb{N}}}{n!c} = (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan\_e}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(10) $0_{\mathbb{C}}\,\mathrm{ExpSeq}$ is absolutely summable and $\sum(0_{\mathbb{C}}\,\mathrm{ExpSeq}) = 1_{\mathbb{C}}$.

Let us consider z. One can verify that $z\,\mathrm{ExpSeq}$ is absolutely summable.

Next we state a number of propositions:

(11) $z\,\mathrm{ExpSeq}(0) = 1_{\mathbb{C}}$ and $(\mathrm{Expan}(0, z, w))(0) = 1_{\mathbb{C}}$.

(12) If $l \leqslant k$, then $(\mathrm{Alfa}(k+1, z, w))(l) = (\mathrm{Alfa}(k, z, w))(l) + (\mathrm{Expan\_e}(k+1, z, w))(l)$.

(13) $(\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(k+1, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(k, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan\_e}(k+1, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(14) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k, z, w))(k)$.

(15) $(\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(16) $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Conj}(k, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(17) $|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k)| \leqslant (\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) \leqslant \sum(|z|\,\mathrm{ExpSeq})$ and $|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k)| \leqslant \sum(|z|\,\mathrm{ExpSeq})$.

(18) $1 \leqslant \sum(|z|\,\mathrm{ExpSeq})$.

(19) $0 \leqslant |z|\,\mathrm{ExpSeq}(n)$.

(20) $|(\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n)$ and if $n \leqslant m$, then $|(\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} |z|\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(21) $|(\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k, z, w)|(\alpha))_{\kappa \in \mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k, z, w)|(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(22) For every p such that $p > 0$ there exists n such that for every k such that $n \leqslant k$ holds $|(\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k, z, w)|(\alpha))_{\kappa \in \mathbb{N}}(k)| < p$.

(23) For every s_1 such that for every k holds $s_1(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Conj}(k, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k)$ holds s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

2. Definition of Exponential Function on Complex

The partial function exp from $\mathbb{C}$ to $\mathbb{C}$ is defined as follows:

(Def. 18) $\mathrm{dom}\exp = \mathbb{C}$ and for every element z of $\mathbb{C}$ holds $(\exp)(z) = \sum(z\,\mathrm{ExpSeq})$.

Let us consider z. The functor $\exp z$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 19) $\exp z = (\exp)(z)$.

The following proposition is true

(24) For all z_1, z_2 holds $\exp z_1 + z_2 = \exp z_1 \cdot \exp z_2$.

3. Definition of Sinus, Cosine, and Exponential Function on $\mathbb{R}$

The partial function sin from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 20) $\operatorname{dom}\sin = \mathbb{R}$ and for every real number d holds $(\sin)(d) = \Im(\sum(0 + d i \operatorname{ExpSeq}))$.

Let us consider t_1. The functor $\sin t_1$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 21) $\sin t_1 = (\sin)(t_1)$.

Next we state the proposition

(25) sin is a function from $\mathbb{R}$ into $\mathbb{R}$.

The partial function cos from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 22) $\operatorname{dom}\cos = \mathbb{R}$ and for every real number d holds $(\cos)(d) = \Re(\sum(0 + d i \operatorname{ExpSeq}))$.

Let us consider t_1. The functor $\cos t_1$ yields an element of $\mathbb{R}$ and is defined by:

(Def. 23) $\cos t_1 = (\cos)(t_1)$.

One can prove the following propositions:

(26) cos is a function from $\mathbb{R}$ into $\mathbb{R}$.

(27) $\operatorname{dom}\sin = \mathbb{R}$ and $\operatorname{dom}\cos = \mathbb{R}$.

(28) $\exp 0 + t_1 i = \cos t_1 + \sin t_1 i$.

(29) $(\exp 0 + t_1 i)^* = \exp -(0 + t_1 i)$.

(30) $|\exp 0 + t_1 i| = 1$ and $|\sin t_1| \leqslant 1$ and $|\cos t_1| \leqslant 1$.

(31) $(\cos)(t_1)^{\mathbf{2}} + (\sin)(t_1)^{\mathbf{2}} = 1$ and $(\cos)(t_1) \cdot (\cos)(t_1) + (\sin)(t_1) \cdot (\sin)(t_1) = 1$.

(32) $(\cos t_1)^{\mathbf{2}} + (\sin t_1)^{\mathbf{2}} = 1$ and $\cos t_1 \cdot \cos t_1 + \sin t_1 \cdot \sin t_1 = 1$.

(33) $(\cos)(0) = 1$ and $(\sin)(0) = 0$ and $(\cos)(-t_1) = (\cos)(t_1)$ and $(\sin)(-t_1) = -(\sin)(t_1)$.

(34) $\cos 0 = 1$ and $\sin 0 = 0$ and $\cos -t_1 = \cos t_1$ and $\sin -t_1 = -\sin t_1$.

Let t_1 be an element of $\mathbb{R}$. The functor t_1 P_sin yielding a sequence of real numbers is defined by:

(Def. 24) For every n holds $t_1\,\mathrm{P\_sin}(n) = \frac{((-1)^n_{\mathbb{N}}) \cdot t_1{}^{2 \cdot n+1}_{\mathbb{N}}}{(2 \cdot n+1)!}$.

Let t_1 be an element of $\mathbb{R}$. The functor t_1 P_cos yielding a sequence of real numbers is defined by:

(Def. 25) For every n holds $t_1\,\mathrm{P\_cos}(n) = \frac{((-1)^n_{\mathbb{N}}) \cdot t_1{}^{2 \cdot n}_{\mathbb{N}}}{(2 \cdot n)!}$.

The following propositions are true:

(35) For all z, k holds $z^{2 \cdot k}_{\mathbb{N}} = (z^k_{\mathbb{N}})^2_{\mathbb{N}}$ and $z^{2 \cdot k}_{\mathbb{N}} = (z^2_{\mathbb{N}})^k_{\mathbb{N}}$.

(36) For all k, t_1 holds $(0 + t_1 i)^{2 \cdot k}_{\mathbb{N}} = ((-1)^k_{\mathbb{N}}) \cdot t_1{}^{2 \cdot k}_{\mathbb{N}} + 0i$ and $(0 + t_1 i)^{2 \cdot k+1}_{\mathbb{N}} = 0 + (((-1)^k_{\mathbb{N}}) \cdot t_1{}^{2 \cdot k+1}_{\mathbb{N}})i$.

(37) For every n holds $n!c = n! + 0i$.

(38) For all t_1, n holds $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_sin}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} \Im(0 + t_1 i\,\mathrm{ExpSeq})(\alpha))_{\kappa\in\mathbb{N}}(2\cdot n+1)$ and $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_cos}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} \Re(0+ t_1 i\,\mathrm{ExpSeq})(\alpha))_{\kappa\in\mathbb{N}}(2\cdot n)$.

(39) For every t_1 holds $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_sin}(\alpha))_{\kappa\in\mathbb{N}}$ is convergent and $\sum(t_1\,\mathrm{P\_sin}) = \Im(\sum(0 + t_1 i\,\mathrm{ExpSeq}))$ and $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_cos}(\alpha))_{\kappa\in\mathbb{N}}$ is convergent and $\sum(t_1\,\mathrm{P\_cos}) = \Re(\sum(0 + t_1 i\,\mathrm{ExpSeq}))$.

(40) For every t_1 holds $(\cos)(t_1) = \sum(t_1\,\mathrm{P\_cos})$ and $(\sin)(t_1) = \sum(t_1\,\mathrm{P\_sin})$.

(41) For all p, t_1, r_1 such that r_1 is convergent and $\lim r_1 = t_1$ and for every n holds $r_1(n) \geqslant p$ holds $t_1 \geqslant p$.

(42) For all n, k, m such that $n < k$ holds $m! > 0$ and $n! \leqslant k!$.

(43) For all t_1, n, k such that $0 \leqslant t_1$ and $t_1 \leqslant 1$ and $n \leqslant k$ holds $t_1{}^{k}_{\mathbb{N}} \leqslant t_1{}^{n}_{\mathbb{N}}$.

(44) For all t_1, n holds $(t_1 + 0i)^{n}_{\mathbb{N}} = (t_1{}^{n}_{\mathbb{N}}) + 0i$.

(45) For all t_1, n holds $\frac{(t_1+0i)^{n}_{\mathbb{N}}}{n!c} = \frac{t_1{}^{n}_{\mathbb{N}}}{n!} + 0i$.

(46) $\Im(\sum(p + 0i\,\mathrm{ExpSeq})) = 0$.

(47) $(\cos)(1) > 0$ and $(\sin)(1) > 0$ and $(\cos)(1) < (\sin)(1)$.

(48) For every t_1 holds $t_1\,\mathrm{ExpSeq} = \Re(t_1 + 0i\,\mathrm{ExpSeq})$.

(49) For every t_1 holds $t_1\,\mathrm{ExpSeq}$ is summable and $\sum(t_1\,\mathrm{ExpSeq}) = \Re(\sum(t_1+ 0i\,\mathrm{ExpSeq}))$.

(50) For all p, q holds $\sum(p + q\,\mathrm{ExpSeq}) = \sum(p\,\mathrm{ExpSeq}) \cdot \sum(q\,\mathrm{ExpSeq})$.

The partial function exp from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 26) $\mathrm{dom}\exp = \mathbb{R}$ and for every real number d holds $(\exp)(d) = \sum(d\,\mathrm{ExpSeq})$.

Let us consider t_1. The functor $\exp t_1$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 27) $\exp t_1 = (\exp)(t_1)$.

We now state a number of propositions:

(51) $\mathrm{dom}\exp = \mathbb{R}$.

(52) For every element d of $\mathbb{R}$ holds $(\exp)(d) = \sum(d\,\mathrm{ExpSeq})$.

(53) For every t_1 holds $(\exp)(t_1) = \Re(\sum(t_1 + 0i\,\mathrm{ExpSeq}))$.

(54) $\exp t_1 + 0i = \exp t_1 + 0i$.

(55) $\exp p + q = \exp p \cdot \exp q$.

(56) $\exp 0 = 1$.

(57) For every t_1 such that $t_1 > 0$ holds $(\exp)(t_1) \geqslant 1$.

(58) For every t_1 such that $t_1 < 0$ holds $0 < (\exp)(t_1)$ and $(\exp)(t_1) \leqslant 1$.

(59) For every t_1 holds $(\exp)(t_1) > 0$.

(60) For every t_1 holds $\exp t_1 > 0$.

4. Differential of Sinus, Cosine, and Exponential Function

Let z be an element of $\mathbb{C}$. The functor $z\,\mathrm{P\_dt}$ yields a complex sequence and is defined as follows:

(Def. 28) For every n holds $z\,\mathrm{P\_dt}(n) = \frac{z_{\mathbb{N}}^{n+1}}{n+2!c}$.

Let z be an element of $\mathbb{C}$. The functor $z\,\mathrm{P\_t}$ yielding a complex sequence is defined by:

(Def. 29) For every n holds $z\,\mathrm{P\_t}(n) = \frac{z_{\mathbb{N}}^{n}}{n+2!c}$.

Next we state a number of propositions:

(61) For every z holds $z\,\mathrm{P\_dt}$ is absolutely summable.

(62) For every z holds $z \cdot \sum(z\,\mathrm{P\_dt}) = \sum(z\,\mathrm{ExpSeq}) - 1_{\mathbb{C}} - z$.

(63) For every p such that $p > 0$ there exists r such that $r > 0$ and for every z such that $|z| < r$ holds $|\sum(z\,\mathrm{P\_dt})| < p$.

(64) For all z, z_1 holds $\sum(z_1 + z\,\mathrm{ExpSeq}) - \sum(z_1\,\mathrm{ExpSeq}) = \sum(z_1\,\mathrm{ExpSeq}) \cdot z + z \cdot \sum(z\,\mathrm{P\_dt}) \cdot \sum(z_1\,\mathrm{ExpSeq})$.

(65) For all p, q holds $(\cos)(p+q) - (\cos)(p) = -q \cdot (\sin)(p) - q \cdot \Im(\sum(0 + qi\,\mathrm{P\_dt}) \cdot ((\cos)(p) + (\sin)(p)i))$.

(66) For all p, q holds $(\sin)(p+q) - (\sin)(p) = q \cdot (\cos)(p) + q \cdot \Re(\sum(0 + qi\,\mathrm{P\_dt}) \cdot ((\cos)(p) + (\sin)(p)i))$.

(67) For all p, q holds $(\exp)(p+q) - (\exp)(p) = q \cdot (\exp)(p) + q \cdot (\exp)(p) \cdot \Re(\sum(q + 0i\,\mathrm{P\_dt}))$.

(68) For every p holds cos is differentiable in p and $(\cos)'(p) = -(\sin)(p)$.

(69) For every p holds sin is differentiable in p and $(\sin)'(p) = (\cos)(p)$.

(70) For every p holds exp is differentiable in p and $(\exp)'(p) = (\exp)(p)$.

(71) exp is differentiable on $\mathbb{R}$ and for every t_1 such that $t_1 \in \mathbb{R}$ holds $(\exp)'(t_1) = (\exp)(t_1)$.

(72) cos is differentiable on $\mathbb{R}$ and for every t_1 such that $t_1 \in \mathbb{R}$ holds $(\cos)'(t_1) = -(\sin)(t_1)$.

(73) sin is differentiable on $\mathbb{R}$ and for every t_1 holds $(\sin)'(t_1) = (\cos)(t_1)$.

(74) For every t_1 such that $t_1 \in [0,1]$ holds $0 < (\cos)(t_1)$ and $(\cos)(t_1) \geqslant \frac{1}{2}$.

(75) $[0,1] \subseteq \mathrm{dom}(\frac{\sin}{\cos})$ and $]0,1[\subseteq \mathrm{dom}(\frac{\sin}{\cos})$.

(76) $\frac{\sin}{\cos}$ is continuous on $[0,1]$.

(77) For all t_2, t_3 such that $t_2 \in]0,1[$ and $t_3 \in]0,1[$ and $(\frac{\sin}{\cos})(t_2) = (\frac{\sin}{\cos})(t_3)$ holds $t_2 = t_3$.

5. Existence of Circle Ratio

The element Pai of $\mathbb{R}$ is defined as follows:

(Def. 30) $(\frac{\sin}{\cos})(\frac{\text{Pai}}{4}) = 1$ and $\text{Pai} \in]0, 4[$.

We now state the proposition

(78) $(\sin)(\frac{\text{Pai}}{4}) = (\cos)(\frac{\text{Pai}}{4})$.

6. Formulas of Sinus, Cosine

Next we state several propositions:

(79) $(\sin)(t_2+t_3) = (\sin)(t_2)\cdot(\cos)(t_3)+(\cos)(t_2)\cdot(\sin)(t_3)$ and $(\cos)(t_2+t_3) = (\cos)(t_2) \cdot (\cos)(t_3) - (\sin)(t_2) \cdot (\sin)(t_3)$.

(80) $\sin t_2 + t_3 = \sin t_2 \cdot \cos t_3 + \cos t_2 \cdot \sin t_3$ and $\cos t_2 + t_3 = \cos t_2 \cdot \cos t_3 - \sin t_2 \cdot \sin t_3$.

(81) $(\cos)(\frac{\text{Pai}}{2}) = 0$ and $(\sin)(\frac{\text{Pai}}{2}) = 1$ and $(\cos)(\text{Pai}) = -1$ and $(\sin)(\text{Pai}) = 0$ and $(\cos)(\text{Pai}+\frac{\text{Pai}}{2}) = 0$ and $(\sin)(\text{Pai}+\frac{\text{Pai}}{2}) = -1$ and $(\cos)(2 \cdot \text{Pai}) = 1$ and $(\sin)(2 \cdot \text{Pai}) = 0$.

(82) $\cos \frac{\text{Pai}}{2} = 0$ and $\sin \frac{\text{Pai}}{2} = 1$ and $\cos \text{Pai} = -1$ and $\sin \text{Pai} = 0$ and $\cos \text{Pai} + \frac{\text{Pai}}{2} = 0$ and $\sin \text{Pai} + \frac{\text{Pai}}{2} = -1$ and $\cos 2 \cdot \text{Pai} = 1$ and $\sin 2 \cdot \text{Pai} = 0$.

(83)(i) $(\sin)(t_1 + 2 \cdot \text{Pai}) = (\sin)(t_1)$,
(ii) $(\cos)(t_1 + 2 \cdot \text{Pai}) = (\cos)(t_1)$,
(iii) $(\sin)(\frac{\text{Pai}}{2} - t_1) = (\cos)(t_1)$,
(iv) $(\cos)(\frac{\text{Pai}}{2} - t_1) = (\sin)(t_1)$,
(v) $(\sin)(\frac{\text{Pai}}{2} + t_1) = (\cos)(t_1)$,
(vi) $(\cos)(\frac{\text{Pai}}{2} + t_1) = -(\sin)(t_1)$,
(vii) $(\sin)(\text{Pai}+t_1) = -(\sin)(t_1)$, and
(viii) $(\cos)(\text{Pai}+t_1) = -(\cos)(t_1)$.

(84) $\sin t_1 + 2 \cdot \text{Pai} = \sin t_1$ and $\cos t_1 + 2 \cdot \text{Pai} = \cos t_1$ and $\sin \frac{\text{Pai}}{2} - t_1 = \cos t_1$ and $\cos \frac{\text{Pai}}{2} - t_1 = \sin t_1$ and $\sin \frac{\text{Pai}}{2} + t_1 = \cos t_1$ and $\cos \frac{\text{Pai}}{2} + t_1 = -\sin t_1$ and $\sin \text{Pai} + t_1 = -\sin t_1$ and $\cos \text{Pai} + t_1 = -\cos t_1$.

(85) For every t_1 such that $t_1 \in]0, \frac{\text{Pai}}{2}[$ holds $(\cos)(t_1) > 0$.

(86) For every t_1 such that $t_1 \in]0, \frac{\text{Pai}}{2}[$ holds $\cos t_1 > 0$.

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