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Kernel Projections and Quotient Lattices

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Summary. This article completes the Mizar formalization of Chapter I, Section 2 from [12]. After presenting some preliminary material (not all of which is later used in this article) we give the proof of theorem 2.7 (i), p.60. We do not follow the hint from [12] suggesting using the equations 2.3, p. 58. The proof is taken directly from the definition of continuous lattice. The goal of the last section is to prove the correspondence between the set of all congruences of a continuous lattice and the set of all kernel operators of the lattice which preserve directed sups (Corollary 2.13).

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The terminology and notation used here are introduced in the following articles: [23], [19], [18], [7], [8], [6], [1], [2], [21], [13], [20], [17], [24], [25], [22], [11], [16], [4], [10], [5], [3], [14], [26], [15], and [9].

1. Preliminaries

The following two propositions are true:

(1) For every set X and for every subset S of $\triangle_X$ holds $\pi_1(S) = \pi_2(S)$.

(2) For all non empty sets X, Y and for every function f from X into Y holds $[\![f, f]\!]^{-1}(\triangle_Y)$ is an equivalence relation of X.

Let L_1, L_2, T_1, T_2 be relational structures, let f be a map from L_1 into T_1, and let g be a map from L_2 into T_2. Then $[\![f, g]\!]$ is a map from $[\![L_1, L_2]\!]$ into $[\![T_1, T_2]\!]$.

One can prove the following propositions:

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(3) For all functions f, g and for every set X holds $\pi_1([\!: f, g :\!]^\circ X) \subseteq f^\circ \pi_1(X)$ and $\pi_2([\!: f, g :\!]^\circ X) \subseteq g^\circ \pi_2(X)$.

(4) For all functions f, g and for every set X such that $X \subseteq [\!: \mathrm{dom}\, f, \mathrm{dom}\, g :\!]$ holds $\pi_1([\!: f, g :\!]^\circ X) = f^\circ \pi_1(X)$ and $\pi_2([\!: f, g :\!]^\circ X) = g^\circ \pi_2(X)$.

(5) For every non empty antisymmetric relational structure S such that inf $\emptyset$ exists in S holds S is upper-bounded.

(6) For every non empty antisymmetric relational structure S such that sup $\emptyset$ exists in S holds S is lower-bounded.

(7) Let L_1, L_2 be antisymmetric non empty relational structures and D be a subset of $[\!: L_1, L_2 :\!]$. If inf D exists in $[\!: L_1, L_2 :\!]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.

(8) Let L_1, L_2 be antisymmetric non empty relational structures and D be a subset of $[\!: L_1, L_2 :\!]$. If sup D exists in $[\!: L_1, L_2 :\!]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(9) Let L_1, L_2, T_1, T_2 be antisymmetric non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is infs-preserving and g is infs-preserving. Then $[\!: f, g :\!]$ is infs-preserving.

(10) Let L_1, L_2, T_1, T_2 be antisymmetric reflexive non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is filtered-infs-preserving and g is filtered-infs-preserving. Then $[\!: f, g :\!]$ is filtered-infs-preserving.

(11) Let L_1, L_2, T_1, T_2 be antisymmetric non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is sups-preserving and g is sups-preserving. Then $[\!: f, g :\!]$ is sups-preserving.

(12) Let L_1, L_2, T_1, T_2 be antisymmetric reflexive non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $[\!: f, g :\!]$ is directed-sups-preserving.

(13) Let L be an antisymmetric non empty relational structure and X be a subset of $[\!: L, L :\!]$. Suppose $X \subseteq \triangle_{\text{the carrier of } L}$ and inf X exists in $[\!: L, L :\!]$. Then $\inf X \in \triangle_{\text{the carrier of } L}$.

(14) Let L be an antisymmetric non empty relational structure and X be a subset of $[\!: L, L :\!]$. Suppose $X \subseteq \triangle_{\text{the carrier of } L}$ and sup X exists in $[\!: L, L :\!]$. Then $\sup X \in \triangle_{\text{the carrier of } L}$.

(15) Let L, M be non empty relational structures. If L and M are isomorphic and L is reflexive, then M is reflexive.

(16) Let L, M be non empty relational structures. If L and M are isomorphic and L is transitive, then M is transitive.

(17) Let L, M be non empty relational structures. Suppose L and M are isomorphic and L is antisymmetric. Then M is antisymmetric.

(18) Let L, M be non empty relational structures. If L and M are isomorphic and L is complete, then M is complete.

(19) Let L be a non empty transitive relational structure and k be a map from L into L. If k is infs-preserving, then k° is infs-preserving.

(20) Let L be a non empty transitive relational structure and k be a map from L into L. If k is filtered-infs-preserving, then k° is filtered-infs-preserving.

(21) Let L be a non empty transitive relational structure and k be a map from L into L. If k is sups-preserving, then k° is sups-preserving.

(22) Let L be a non empty transitive relational structure and k be a map from L into L. If k is directed-sups-preserving, then k° is directed-sups-preserving.

(23) Let S, T be reflexive antisymmetric non empty relational structures and f be a map from S into T. If f is directed-sups-preserving, then f is monotone.

(24) Let S, T be reflexive antisymmetric non empty relational structures and f be a map from S into T. If f is filtered-infs-preserving, then f is monotone.

(25) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is monotone. Let X be a subset of S. If X is filtered, then $f^\circ X$ is filtered.

(26) Let L_1, L_2, L_3 be non empty relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is infs-preserving and g is infs-preserving. Then $g \cdot f$ is infs-preserving.

(27) Let L_1, L_2, L_3 be non empty reflexive antisymmetric relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is filtered-infs-preserving and g is filtered-infs-preserving. Then $g \cdot f$ is filtered-infs-preserving.

(28) Let L_1, L_2, L_3 be non empty relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is sups-preserving and g is sups-preserving. Then $g \cdot f$ is sups-preserving.

(29) Let L_1, L_2, L_3 be non empty reflexive antisymmetric relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

2. Some Remarks on Lattice Product

We now state several propositions:

(30) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a lower-bounded antisymmetric relational structure. Then $\prod J$ is lower-bounded.

(31) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an upper-bounded antisymmetric relational structure. Then $\prod J$ is upper-bounded.

(32) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a lower-bounded antisymmetric relational structure. Let i be an element of I. Then $\perp_{\prod J}(i) = \perp_{J(i)}$.

(33) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an upper-bounded antisymmetric relational structure. Let i be an element of I. Then $\top_{\prod J}(i) = \top_{J(i)}$.

(34) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a continuous complete lattice. Then $\prod J$ is continuous.

3. Kernel Projections and Quotient Lattices

We now state the proposition

(35) Let L, T be continuous complete lattices, g be a CLHomomorphism of L, T, and S be a subset of the carrier of $[\!: L, L :\!]$. Suppose $S = [\!: g, g :\!]^{-1}(\triangle_{\text{the carrier of } T})$. Then $\text{sub}(S)$ is a continuous subframe of $[\!: L, L :\!]$.

Let L be a relational structure and let R be a subset of the carrier of $[\!: L, L :\!]$. Let us assume that R is an equivalence relation of the carrier of L. The functor $\text{EqRel}(R)$ yields an equivalence relation of the carrier of L and is defined by:

(Def. 1) $\text{EqRel}(R) = R$.

Let L be a non empty relational structure and let R be a subset of $[\!: L, L :\!]$. We say that R is a continuous lattice congruence if and only if:

(Def. 2) R is an equivalence relation of the carrier of L and $\text{sub}(R)$ is a continuous subframe of $[\!: L, L :\!]$.

We now state the proposition

(36) Let L be a complete lattice and R be a non empty subset of $[: L, L :]$. Suppose R is a continuous lattice congruence. Let x be an element of the carrier of L. Then $\langle \inf([x]_{\mathrm{EqRel}(R)}), x \rangle \in R$.

Let L be a complete lattice and let R be a non empty subset of $[: L, L :]$. Let us assume that R is a continuous lattice congruence. The kernel operation of R yields a kernel map from L into L and is defined by:

(Def. 3) For every element x of L holds (the kernel operation of R)$(x) = \inf([x]_{\mathrm{EqRel}(R)})$.

Next we state three propositions:

(37) Let L be a complete lattice and R be a non empty subset of $[: L, L :]$. Suppose R is a continuous lattice congruence. Then

(i) the kernel operation of R is directed-sups-preserving, and

(ii) $R = [:$ the kernel operation of R, the kernel operation of $R :]^{-1}(\triangle_{\text{the carrier of } L})$.

(38) Let L be a continuous complete lattice, R be a subset of $[: L, L :]$, and k be a kernel map from L into L. Suppose k is directed-sups-preserving and $R = [: k, k :]^{-1}(\triangle_{\text{the carrier of } L})$. Then there exists a continuous complete strict lattice L_4 such that

(i) the carrier of $L_4 = \mathrm{Classes}\,\mathrm{EqRel}(R)$,

(ii) the internal relation of $L_4 = \{\langle [x]_{\mathrm{EqRel}(R)}, [y]_{\mathrm{EqRel}(R)} \rangle; x$ ranges over elements of L, y ranges over elements of L: $k(x) \leqslant k(y)\}$, and

(iii) for every map g from L into L_4 such that for every element x of L holds $g(x) = [x]_{\mathrm{EqRel}(R)}$ holds g is a CLHomomorphism of L, L_4.

(39) Let L be a continuous complete lattice and R be a subset of $[: L, L :]$. Suppose that

(i) R is an equivalence relation of the carrier of L, and

(ii) there exists a continuous complete lattice L_4 such that the carrier of $L_4 = \mathrm{Classes}\,\mathrm{EqRel}(R)$ and for every map g from L into L_4 such that for every element x of L holds $g(x) = [x]_{\mathrm{EqRel}(R)}$ holds g is a CLHomomorphism of L, L_4.

Then $\mathrm{sub}(R)$ is a continuous subframe of $[: L, L :]$.

Let L be a non empty reflexive relational structure. Observe that there exists a map from L into L which is directed-sups-preserving and kernel.

Let L be a non empty reflexive relational structure and let k be a kernel map from L into L. The kernel congruence of k yields a non empty subset of $[: L, L :]$ and is defined by:

(Def. 4) The kernel congruence of $k = [: k, k :]^{-1}(\triangle_{\text{the carrier of } L})$.

We now state two propositions:

(40) Let L be a non empty reflexive relational structure and k be a kernel map from L into L. Then the kernel congruence of k is an equivalence relation of the carrier of L.

(41) Let L be a continuous complete lattice and k be a directed-sups-preserving kernel map from L into L. Then the kernel congruence of k is a continuous lattice congruence.

Let L be a continuous complete lattice and let R be a non empty subset of $[: L, L :]$. Let us assume that R is a continuous lattice congruence. The functor ${}^{L}/_{R}$ yielding a continuous complete strict lattice is defined by:

(Def. 5) The carrier of ${}^{L}/_{R}$ = Classes EqRel(R) and for all elements x, y of ${}^{L}/_{R}$ holds $x \leqslant y$ iff $\bigsqcap_L x \leqslant \bigsqcap_L y$.

The following propositions are true:

(42) Let L be a continuous complete lattice and R be a non empty subset of $[: L, L :]$. Suppose R is a continuous lattice congruence. Let x be a set. Then x is an element of ${}^{L}/_{R}$ if and only if there exists an element y of L such that $x = [y]_{\mathrm{EqRel}(R)}$.

(43) Let L be a continuous complete lattice and R be a non empty subset of $[: L, L :]$. Suppose R is a continuous lattice congruence. Then R = the kernel congruence of the kernel operation of R.

(44) Let L be a continuous complete lattice and k be a directed-sups-preserving kernel map from L into L. Then k = the kernel operation of the kernel congruence of k.

(45) Let L be a continuous complete lattice and p be a projection map from L into L. Suppose p is infs-preserving. Then $\mathrm{Im}\, p$ is a continuous lattice and $\mathrm{Im}\, p$ is infs-inheriting.

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