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Representation Theorem for Free Continuous Lattices

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Summary. We present the Mizar formalization of theorem 4.17, Chapter I from [11]: a free continuous lattice with m generators is isomorphic to the lattice of filters of 2^X ($\overline{\overline{X}} = m$) which is freely generated by $\{\uparrow x : x \in X\}$ (the set of ultrafilters).

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The papers [1], [6], [7], [15], [2], [17], [12], [10], [19], [20], [18], [16], [9], [14], [4], [8], [5], [3], and [13] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For every upper-bounded semilattice L and for every non empty directed subset F of $\langle \mathrm{Filt}(L), \subseteq \rangle$ holds $\sup F = \bigcup F$.

(2) Let L, S, T be complete non empty posets, f be a CLHomomorphism of L, S, and g be a CLHomomorphism of S, T. Then $g \cdot f$ is a CLHomomorphism of L, T.

(3) For every non empty relational structure L holds id_L is infs-preserving.

(4) For every non empty relational structure L holds id_L is directed-sups-preserving.

(5) For every complete non empty poset L holds id_L is a CLHomomorphism of L, L.

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(6) For every upper-bounded non empty poset L with g.l.b.'s holds $\langle \mathrm{Filt}(L), \subseteq \rangle$ is a continuous subframe of $2_{\subseteq}^{\text{the carrier of } L}$.

Let L be an upper-bounded non empty poset with g.l.b.'s. Observe that $\langle \mathrm{Filt}(L), \subseteq \rangle$ is continuous.

Let L be an upper-bounded non empty poset. One can check that every element of the carrier of $\langle \mathrm{Filt}(L), \subseteq \rangle$ is non empty.

2. Free Generators of Continuous Lattices

Let S be a continuous complete non empty poset and let A be a set. We say that A is a set of free generators of S if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let T be a continuous complete non empty poset and f be a function from A into the carrier of T. Then there exists a CLHomomorphism h of S, T such that $h{\restriction}A = f$ and for every CLHomomorphism h' of S, T such that $h'{\restriction}A = f$ holds $h' = h$.

Next we state two propositions:

(7) Let S be a continuous complete non empty poset and A be a set. If A is a set of free generators of S, then A is a subset of S.

(8) Let S be a continuous complete non empty poset and A be a set. Suppose A is a set of free generators of S. Let h' be a CLHomomorphism of S, S. If $h'{\restriction}A = \mathrm{id}_A$, then $h' = \mathrm{id}_S$.

3. Representation Theorem for Free Continuous Lattices

In the sequel X is a set, F is a filter of $2_{\subseteq}^{X}$, x is an element of $2_{\subseteq}^{X}$, and z is an element of X.

Let us consider X. The fixed ultrafilters of X is a family of subsets of $2_{\subseteq}^{X}$ and is defined as follows:

(Def. 2) The fixed ultrafilters of $X = \{\uparrow x : \bigvee_z\ x = \{z\}\}$.

One can prove the following three propositions:

(9) The fixed ultrafilters of $X \subseteq \mathrm{Filt}(2_{\subseteq}^{X})$.

(10) $\overline{\overline{\text{the fixed ultrafilters of } X}} = \overline{\overline{X}}$.

(11) $F = \bigsqcup_{(\langle \mathrm{Filt}(2_{\subseteq}^{X}), \subseteq \rangle)} \{\bigsqcap_{(\langle \mathrm{Filt}(2_{\subseteq}^{X}), \subseteq \rangle)} \{\uparrow x : \bigvee_z\ (x = \{z\} \wedge z \in Y)\}; Y$ ranges over subsets of X: $Y \in F\}$.

Let us consider X, let L be a continuous complete non empty poset, and let f be a function from the fixed ultrafilters of X into the carrier of L. The extension of f to homomorphism is a map from $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle$ into L and is defined by the condition (Def. 3).

(Def. 3) Let F_1 be an element of the carrier of $(\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle)$. Then (the extension of f to homomorphism)$(F_1) = \bigsqcup_L \{ \bigsqcap_L \{ f(\uparrow x) : \bigvee_z \ (x = \{z\} \ \wedge \ z \in Y)\}; Y$ ranges over subsets of X: $Y \in F_1\}$.

One can prove the following propositions:

(12) Let L be a continuous complete non empty poset and f be a function from the fixed ultrafilters of X into the carrier of L. Then the extension of f to homomorphism is monotone.

(13) Let L be a continuous complete non empty poset and f be a function from the fixed ultrafilters of X into the carrier of L. Then (the extension of f to homomorphism)$(\top_{\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle}) = \top_L$.

Let us consider X, let L be a continuous complete non empty poset, and let f be a function from the fixed ultrafilters of X into the carrier of L. Observe that the extension of f to homomorphism is directed-sups-preserving.

Let us consider X, let L be a continuous complete non empty poset, and let f be a function from the fixed ultrafilters of X into the carrier of L. Note that the extension of f to homomorphism is infs-preserving.

The following propositions are true:

(14) Let L be a continuous complete non empty poset and f be a function from the fixed ultrafilters of X into the carrier of L. Then (the extension of f to homomorphism)$\restriction$(the fixed ultrafilters of X) $= f$.

(15) Let L be a continuous complete non empty poset, f be a function from the fixed ultrafilters of X into the carrier of L, and h be a CLHomomorphism of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle$, L. Suppose $h\restriction$the fixed ultrafilters of $X = f$. Then $h =$ the extension of f to homomorphism.

(16) The fixed ultrafilters of X is a set of free generators of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle$.

(17) Let L, M be continuous complete lattices and F, G be sets. Suppose F is a set of free generators of L and G is a set of free generators of M and $\overline{\overline{F}} = \overline{\overline{G}}$. Then L and M are isomorphic.

(18) Let L be a continuous complete lattice and G be a set. Suppose G is a set of free generators of L and $\overline{\overline{G}} = \overline{\overline{X}}$. Then L and $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq\rangle$ are isomorphic.

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