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Defining by Structural Induction in the Positive Propositional Language

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Summary. The main goal of the paper consists in proving schemes for defining by structural induction in the language defined by Adam Grabowski [13]. The article consists of four parts. Besides the preliminaries where we prove some simple facts still missing in the library, they are:

- "About the language" in which the consequences of the fact that the algebra of formulae is free are formulated,

- "Defining by structural induction" in which two schemes are proved,

- "The tree of the subformulae" in which a scheme proved in the previous section is used to define the tree of subformulae; also some simple facts about the tree are proved.

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The terminology and notation used in this paper are introduced in the following papers: [16], [19], [1], [14], [20], [10], [12], [18], [8], [15], [9], [11], [3], [17], [2], [4], [5], [6], [7], and [13].

1. Preliminaries

In this paper X, x denote sets.

We now state four propositions:

(1) Let Z be a set and M be a many sorted set indexed by Z. Suppose that for every set x such that $x \in Z$ holds $M(x)$ is a many sorted set indexed by x. Let f be a function. If $f = \mathrm{Union}\, M$, then $\mathrm{dom}\, f = \bigcup Z$.

(2) For all sets x, y and for all finite sequences f, g such that $\langle x \rangle ^\frown f = \langle y \rangle ^\frown g$ holds $f = g$.

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(3) If $\langle x \rangle$ is a finite sequence of elements of X, then $x \in X$.

(4) Let given X and f be a finite sequence of elements of X. Suppose $f \neq \varepsilon$. Then there exists a finite sequence g of elements of X and there exists an element d of X such that $f = g \frown \langle d \rangle$.

We adopt the following rules: m, n are natural numbers, p, q, r, s are elements of HP-WFF, and T_1, T_2 are trees.

Next we state the proposition

(5) $\langle x \rangle \in \overbrace{T_1, T_2}$ iff $x = 0$ or $x = 1$.

Let us mention that ε is tree yielding.

The scheme *InTreeInd* deals with a tree $\mathcal{A}$ and and states that:

For every element f of $\mathcal{A}$ holds $\mathcal{P}[f]$

provided the following conditions are satisfied:

- $\mathcal{P}[\varepsilon_{\mathbb{N}}]$, and
- For every element f of $\mathcal{A}$ such that $\mathcal{P}[f]$ and for every n such that $f \frown \langle n \rangle \in \mathcal{A}$ holds $\mathcal{P}[f \frown \langle n \rangle]$.

In the sequel D is a non empty set and T_1, T_2 are decorated trees.

Next we state three propositions:

(6) For every set x and for all T_1, T_2 holds $(x\text{-tree}(T_1, T_2))(\varepsilon) = x$.

(7) $(x\text{-tree}(T_1, T_2))(\langle 0 \rangle) = T_1(\varepsilon)$ and $(x\text{-tree}(T_1, T_2))(\langle 1 \rangle) = T_2(\varepsilon)$.

(8) $(x\text{-tree}(T_1, T_2)) \restriction \langle 0 \rangle = T_1$ and $(x\text{-tree}(T_1, T_2)) \restriction \langle 1 \rangle = T_2$.

Let us consider x and let p be a decorated tree yielding non empty finite sequence. Observe that x-tree(p) is non root.

Let us consider x and let T_1 be a decorated tree. Observe that x-tree(T_1) is non root. Let T_2 be a decorated tree. Observe that x-tree(T_1, T_2) is non root.

2. About the Language

Let us consider n. The functor prop n yielding an element of HP-WFF is defined as follows:

(Def. 1) $\text{prop}\, n = \langle 3 + n \rangle$.

Let D be a set. Let us observe that D has VERUM if and only if:

(Def. 2) VERUM $\in D$.

Let us observe that D has propositional variables if and only if:

(Def. 3) For every n holds $\text{prop}\, n \in D$.

Let D be a subset of HP-WFF. Let us observe that D has implication if and only if:

(Def. 4) For all p, q such that $p \in D$ and $q \in D$ holds $p \Rightarrow q \in D$.

Let us observe that D has conjunction if and only if:

(Def. 5) For all p, q such that $p \in D$ and $q \in D$ holds $p \wedge q \in D$.

In the sequel t denotes a finite sequence.

Let us consider p. We say that p is conjunctive if and only if:

(Def. 6) There exist r, s such that $p = r \wedge s$.

We say that p is conditional if and only if:

(Def. 7) There exist r, s such that $p = r \Rightarrow s$.

We say that p is simple if and only if:

(Def. 8) There exists n such that $p = \mathrm{prop}\, n$.

The scheme *HP Ind* concerns and states that:

For every r holds $\mathcal{P}[r]$

provided the following requirements are met:

- $\mathcal{P}[\mathrm{VERUM}]$,
- For every n holds $\mathcal{P}[\mathrm{prop}\, n]$, and
- For all r, s such that $\mathcal{P}[r]$ and $\mathcal{P}[s]$ holds $\mathcal{P}[r \wedge s]$ and $\mathcal{P}[r \Rightarrow s]$.

Next we state a number of propositions:

(9) p is conjunctive, or conditional, or simple or $p = \mathrm{VERUM}$.

(10) $\mathrm{len}\, p \geqslant 1$.

(11) If $p(1) = 1$, then p is conditional.

(12) If $p(1) = 2$, then p is conjunctive.

(13) If $p(1) = 3 + n$, then p is simple.

(14) If $p(1) = 0$, then $p = \mathrm{VERUM}$.

(15) $\mathrm{len}\, p < \mathrm{len}(p \wedge q)$ and $\mathrm{len}\, q < \mathrm{len}(p \wedge q)$.

(16) $\mathrm{len}\, p < \mathrm{len}(p \Rightarrow q)$ and $\mathrm{len}\, q < \mathrm{len}(p \Rightarrow q)$.

(17) If $p = q \frown t$, then $p = q$.

(18) If $p \frown q = r \frown s$, then $p = r$ and $q = s$.

(19) If $p \wedge q = r \wedge s$, then $p = r$ and $s = q$.

(20) If $p \Rightarrow q = r \Rightarrow s$, then $p = r$ and $s = q$.

(21) If $\mathrm{prop}\, n = \mathrm{prop}\, m$, then $n = m$.

(22) $p \wedge q \neq r \Rightarrow s$.

(23) $p \wedge q \neq \mathrm{VERUM}$.

(24) $p \wedge q \neq \mathrm{prop}\, n$.

(25) $p \Rightarrow q \neq \mathrm{VERUM}$.

(26) $p \Rightarrow q \neq \mathrm{prop}\, n$.

(27) $p \wedge q \neq p$ and $p \wedge q \neq q$.

(28) $p \Rightarrow q \neq p$ and $p \Rightarrow q \neq q$.

(29) $\mathrm{VERUM} \neq \mathrm{prop}\, n$.

3. Defining by Structural Induction

Now we present two schemes. The scheme *HP MSSExL* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a 5-ary predicate $\mathcal{Q}$, and states that:

There exists a many sorted set M indexed by HP-WFF such that

(i) $M(\text{VERUM}) = \mathcal{A}$,

(ii) for every n holds $M(\text{prop}\, n) = \mathcal{F}(n)$, and

(iii) for all p, q and for all sets a, b, c, d such that $a = M(p)$ and $b = M(q)$ and $c = M(p \wedge q)$ and $d = M(p \Rightarrow q)$ holds $\mathcal{P}[p, q, a, b, c]$ and $\mathcal{Q}[p, q, a, b, d]$

provided the following conditions are met:

- For all p, q and for all sets a, b there exists a set c such that $\mathcal{P}[p, q, a, b, c]$,
- For all p, q and for all sets a, b there exists a set d such that $\mathcal{Q}[p, q, a, b, d]$,
- For all p, q and for all sets a, b, c, d such that $\mathcal{P}[p, q, a, b, c]$ and $\mathcal{P}[p, q, a, b, d]$ holds $c = d$, and
- For all p, q and for all sets a, b, c, d such that $\mathcal{Q}[p, q, a, b, c]$ and $\mathcal{Q}[p, q, a, b, d]$ holds $c = d$.

The scheme *HP MSSLambda* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two binary functors $\mathcal{G}$ and $\mathcal{H}$ yielding sets, and states that:

There exists a many sorted set M indexed by HP-WFF such that

(i) $M(\text{VERUM}) = \mathcal{A}$,

(ii) for every n holds $M(\text{prop}\, n) = \mathcal{F}(n)$, and

(iii) for all p, q and for all sets x, y such that $x = M(p)$ and $y = M(q)$ holds $M(p \wedge q) = \mathcal{G}(x, y)$ and $M(p \Rightarrow q) = \mathcal{H}(x, y)$

for all values of the parameters.

4. The Tree of the Subformulae

The many sorted set HP-Subformulae indexed by HP-WFF is defined by the conditions (Def. 9).

(Def. 9)(i) (HP-Subformulae)(VERUM) = the root tree of VERUM,

(ii) for every n holds (HP-Subformulae)(prop n) = the root tree of prop n, and

(iii) for all p, q there exist trees p', q' decorated with elements of HP-WFF such that $p' =$ (HP-Subformulae)(p) and $q' =$ (HP-Subformulae)(q) and (HP-Subformulae)($p \wedge q$) $= p \wedge q$-tree(p', q') and (HP-Subformulae)($p \Rightarrow q$) $= (p \Rightarrow q)$-tree(p', q').

Let us consider p. The functor Subformulae p yielding a tree decorated with elements of HP-WFF is defined by:

(Def. 10) Subformulae p = (HP-Subformulae)(p).

The following propositions are true:

(30) Subformulae VERUM = the root tree of VERUM.

(31) Subformulae prop n = the root tree of prop n.

(32) Subformulae($p \wedge q$) = $p \wedge q$-tree(Subformulae p, Subformulae q).

(33) Subformulae($p \Rightarrow q$) = ($p \Rightarrow q$)-tree(Subformulae p, Subformulae q).

(34) (Subformulae p)(ε) = p.

(35) For every element f of dom Subformulae p holds Subformulae $p \restriction f$ = Subformulae(Subformulae p)(f).

(36) If $p \in$ Leaves(Subformulae q), then p = VERUM or p is simple.

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