FORMALIZED MATHEMATICS
Volume 8, Number 1, 1999
University of Białystok

Rotating and Reversing

Andrzej Trybulec
University of Białystok

Summary. Quite a number of lemmas for the Jordan curve theorem, as yet in the case of the special polygonal curves, have been proved. By "special" we mean, that it is a polygonal curve with edges parallel to axes and actually the lemmas have been proved, mostly, for the triangulations i.e. for finite sequences that define the curve. Moreover some of the results deal only with a special case:

- finite sequences are clockwise oriented,
- the first member of the sequence is the member with the lowest ordinate among those with the highest abscissa (N-min f, where f is a finite sequence, in the Mizar jargon).

In the change of the orientation one has to reverse the sequence (the operation introduced in [7]) and to change the second restriction one has to rotate the sequence (the operation introduced in [26]). The goal of the paper is to prove, mostly simple, facts about the relationship between properties and attributes of the finite sequence and its rotation (similar results about reversing had been proved in [7]). Some of them deal with recounting parameters, others with properties that are invariant under the rotation. We prove also that the finite sequence is either clockwise oriented or it is such after reversing. Everything is proved for the so called standard finite sequences, which means that if a point belongs to it then every point with the same abscissa or with the same ordinate, that belongs to the polygon, belongs also to the finite sequence. It does not seem that this requirement causes serious technical obstacles.

MML Identifier: `REVROT_1`.

The terminology and notation used here are introduced in the following articles: [24], [29], [12], [2], [23], [20], [1], [4], [6], [3], [5], [13], [28], [14], [7], [26], [22], [30], [21], [9], [10], [11], [15], [16], [18], [25], [8], [17], [27], and [19].

ISSN 1426–2630

1. Preliminaries

For simplicity, we use the following convention: i, k, m, n are natural numbers, D is a non empty set, p is an element of D, and f is a finite sequence of elements of D.

Let S be a non trivial 1-sorted structure. Observe that the carrier of S is non trivial.

Let D be a non empty set and let f be a finite sequence of elements of D. Let us observe that f is constant if and only if:

(Def. 1) For all n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ holds $\pi_n f = \pi_m f$.

One can prove the following three propositions:

(1) Let D be a non empty set and f be a finite sequence of elements of D. If f yields $\pi_{\operatorname{len} f} f$ just once, then $(\pi_{\operatorname{len} f} f) \hookleftarrow f = \operatorname{len} f$.

(2) For every non empty set D and for every finite sequence f of elements of D holds $f_{\lfloor \operatorname{len} f} = \emptyset$.

(3) For every non empty set D and for every non empty finite sequence f of elements of D holds $\pi_{\operatorname{len} f} f \in \operatorname{rng} f$.

Let D be a non empty set, let f be a finite sequence of elements of D, and let y be a set. Let us observe that f yields y just once if and only if:

(Def. 2) There exists a set x such that $x \in \operatorname{dom} f$ and $y = \pi_x f$ and for every set z such that $z \in \operatorname{dom} f$ and $z \neq x$ holds $\pi_z f \neq y$.

The following propositions are true:

(4) Let D be a non empty set and f be a finite sequence of elements of D. If f yields $\pi_{\operatorname{len} f} f$ just once, then $f -: \pi_{\operatorname{len} f} f = f$.

(5) Let D be a non empty set and f be a finite sequence of elements of D. If f yields $\pi_{\operatorname{len} f} f$ just once, then $f :- \pi_{\operatorname{len} f} f = \langle \pi_{\operatorname{len} f} f \rangle$.

(6) $1 \leqslant \operatorname{len}(f :- p)$.

(7) Let D be a non empty set, p be an element of D, and f be a finite sequence of elements of D. If $p \in \operatorname{rng} f$, then $\operatorname{len}(f :- p) \leqslant \operatorname{len} f$.

(8) For every non empty set D and for every circular non empty finite sequence f of elements of D holds $\operatorname{Rev}(f)$ is circular.

2. About the Rotation

In the sequel D denotes a non empty set, p denotes an element of D, and f denotes a finite sequence of elements of D.

We now state several propositions:

(9) If $p \in \operatorname{rng} f$ and $1 \leqslant i$ and $i \leqslant \operatorname{len}(f :- p)$, then $\pi_i f^{p}_{\circlearrowleft} = \pi_{(i -' 1) + p \looparrowleft f} f$.

(10) If $p \in \operatorname{rng} f$ and $p \looparrowleft f \leqslant i$ and $i \leqslant \operatorname{len} f$, then $\pi_i f = \pi_{(i+1) -' p \looparrowleft f} f^{p}_{\circlearrowleft}$.

(11) If $p \in \operatorname{rng} f$, then $\pi_{\operatorname{len}(f :- p)} f^{p}_{\circlearrowleft} = \pi_{\operatorname{len} f} f$.

(12) If $p \in \operatorname{rng} f$ and $\operatorname{len}(f :- p) < i$ and $i \leqslant \operatorname{len} f$, then $\pi_i f^{p}_{\circlearrowleft} = \pi_{(i + p \looparrowleft f) -' \operatorname{len} f} f$.

(13) If $p \in \operatorname{rng} f$ and $1 < i$ and $i \leqslant p \looparrowleft f$, then $\pi_i f = \pi_{(i + \operatorname{len} f) -' p \looparrowleft f} f^{p}_{\circlearrowleft}$.

(14) $\operatorname{len}(f^{p}_{\circlearrowleft}) = \operatorname{len} f$.

(15) $\operatorname{dom}(f^{p}_{\circlearrowleft}) = \operatorname{dom} f$.

(16) Let D be a non empty set, f be a circular finite sequence of elements of D, and p be an element of D. If for every i such that $1 < i$ and $i < \operatorname{len} f$ holds $\pi_i f \neq \pi_1 f$, then $(f^{p}_{\circlearrowleft})^{\pi_1 f}_{\circlearrowleft} = f$.

3. Rotating Circular Ones

In the sequel f is a circular finite sequence of elements of D.

The following propositions are true:

(17) If $p \in \operatorname{rng} f$ and $\operatorname{len}(f :- p) \leqslant i$ and $i \leqslant \operatorname{len} f$, then $\pi_i f^{p}_{\circlearrowleft} = \pi_{(i + p \looparrowleft f) -' \operatorname{len} f} f$.

(18) If $p \in \operatorname{rng} f$ and $1 \leqslant i$ and $i \leqslant p \looparrowleft f$, then $\pi_i f = \pi_{(i + \operatorname{len} f) -' p \looparrowleft f} f^{p}_{\circlearrowleft}$.

Let D be a non trivial set. Note that there exists a finite sequence of elements of D which is non constant and circular.

Let D be a non trivial set, let p be an element of D, and let f be a non constant circular finite sequence of elements of D. Note that $f^{p}_{\circlearrowleft}$ is non constant.

4. Finite Sequence on the Plane

The following proposition is true

(19) For every non empty natural number n holds $0_{\mathcal{E}^{n}_{\mathrm{T}}} \neq 1.\mathrm{REAL}\, n$.

Let n be a non empty natural number. Note that $\mathcal{E}^{n}_{\mathrm{T}}$ is non trivial.

In the sequel f, g are finite sequences of elements of $\mathcal{E}^{2}_{\mathrm{T}}$.

Next we state four propositions:

(20) If $\operatorname{rng} f \subseteq \operatorname{rng} g$, then $\operatorname{rng} \mathbf{X}\text{-coordinate}(f) \subseteq \operatorname{rng} \mathbf{X}\text{-coordinate}(g)$.

(21) If $\operatorname{rng} f = \operatorname{rng} g$, then $\operatorname{rng} \mathbf{X}\text{-coordinate}(f) = \operatorname{rng} \mathbf{X}\text{-coordinate}(g)$.

(22) If $\operatorname{rng} f \subseteq \operatorname{rng} g$, then $\operatorname{rng} \mathbf{Y}\text{-coordinate}(f) \subseteq \operatorname{rng} \mathbf{Y}\text{-coordinate}(g)$.

(23) If $\operatorname{rng} f = \operatorname{rng} g$, then $\operatorname{rng} \mathbf{Y}\text{-coordinate}(f) = \operatorname{rng} \mathbf{Y}\text{-coordinate}(g)$.

5. Rotating Finite Sequence on the Plane

In the sequel p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$ and f denotes a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let f be a special circular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Observe that $f^p_{\circlearrowleft}$ is special.

The following propositions are true:

(24) If $p \in \operatorname{rng} f$ and $1 \leqslant i$ and $i < \operatorname{len}(f :- p)$, then $\mathcal{L}(f^p_{\circlearrowleft}, i) = \mathcal{L}(f, (i -' 1) + p \leftarrowtail f)$.

(25) If $p \in \operatorname{rng} f$ and $p \leftarrowtail f \leqslant i$ and $i < \operatorname{len} f$, then $\mathcal{L}(f, i) = \mathcal{L}(f^p_{\circlearrowleft}, (i -' p \leftarrowtail f) + 1)$.

(26) For every circular finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{Inc}(\mathbf{X}\text{-coordinate}(f)) = \operatorname{Inc}(\mathbf{X}\text{-coordinate}(f^p_{\circlearrowleft}))$.

(27) For every circular finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{Inc}(\mathbf{Y}\text{-coordinate}(f)) = \operatorname{Inc}(\mathbf{Y}\text{-coordinate}(f^p_{\circlearrowleft}))$.

(28) For every non empty circular finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds the Go-board of $f^p_{\circlearrowleft}$ = the Go-board of f.

(29) For every non constant standard special circular sequence f holds $\operatorname{Rev}(f^p_{\circlearrowleft}) = (\operatorname{Rev}(f))^p_{\circlearrowleft}$.

6. Rotating Circular Ones (on the Plane)

In the sequel f is a circular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

We now state two propositions:

(30) For every circular s.c.c. finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $\operatorname{len} f > 4$ holds $\mathcal{L}(f, \operatorname{len} f -' 1) \cap \mathcal{L}(f, 1) = \{\pi_1 f\}$.

(31) If $p \in \operatorname{rng} f$ and $\operatorname{len}(f{:-}p) \leqslant i$ and $i < \operatorname{len} f$, then $\mathcal{L}(f^p_{\circlearrowleft}, i) = \mathcal{L}(f, (i{+}p \leftarrowtail f) -' \operatorname{len} f)$.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let f be a circular s.c.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. One can check that $f^p_{\circlearrowleft}$ is s.c.c..

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let f be a non constant standard special circular sequence. Observe that $f^p_{\circlearrowleft}$ is unfolded.

Next we state three propositions:

(32) If $p \in \operatorname{rng} f$ and $1 \leqslant i$ and $i < p \leftarrowtail f$, then $\mathcal{L}(f, i) = \mathcal{L}(f^p_{\circlearrowleft}, (i + \operatorname{len} f) -' p \leftarrowtail f)$.

(33) $\widetilde{\mathcal{L}}(f^p_{\circlearrowleft}) = \widetilde{\mathcal{L}}(f)$.

(34) Let G be a Go-board. Then f is a sequence which elements belong to G if and only if $f^p_{\circlearrowleft}$ is a sequence which elements belong to G.

Let p be a point of $\mathcal{E}^2_{\rm T}$ and let f be a standard non empty circular finite sequence of elements of $\mathcal{E}^2_{\rm T}$. One can verify that $f^{p}_{\circlearrowleft}$ is standard.

One can prove the following three propositions:

(35) Let f be a non constant standard special circular sequence and given p, k. If $p \in \mathrm{rng}\, f$ and $1 \leqslant k$ and $k < p \hookleftarrow f$, then $\mathrm{leftcell}(f, k) = \mathrm{leftcell}(f^{p}_{\circlearrowleft}, (k + \mathrm{len}\, f) -' p \hookleftarrow f)$.

(36) For every non constant standard special circular sequence f holds $\mathrm{LeftComp}(f^{p}_{\circlearrowleft}) = \mathrm{LeftComp}(f)$.

(37) For every non constant standard special circular sequence f holds $\mathrm{RightComp}(f^{p}_{\circlearrowleft}) = \mathrm{RightComp}(f)$.

7. The Orientation

Let p be a point of $\mathcal{E}^2_{\rm T}$ and let f be a clockwise oriented non constant standard special circular sequence. One can verify that $f^{p}_{\circlearrowleft}$ is clockwise oriented.

One can prove the following proposition

(38) Let f be a non constant standard special circular sequence. Then f is clockwise oriented or $\mathrm{Rev}(f)$ is clockwise oriented.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.

[8] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[9] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[13] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[14] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[15] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[16] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part II. *Formalized Mathematics*, 3(**1**):117–121, 1992.
[17] Yatsuka Nakamura and Adam Grabowski. Bounding boxes for special sequences in $\mathcal{E}^2$. *Formalized Mathematics*, 7(**1**):115–121, 1998.
[18] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.
[19] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[20] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[21] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[22] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[23] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[24] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[25] Andrzej Trybulec. Left and right component of the complement of a special closed curve. *Formalized Mathematics*, 5(**4**):465–468, 1996.
[26] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(**3**):317–322, 1996.
[27] Andrzej Trybulec and Yatsuka Nakamura. On the order on a special polygon. *Formalized Mathematics*, 6(**4**):541–548, 1997.
[28] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[29] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[30] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received January 21, 1999