FORMALIZED MATHEMATICS
Volume 8, Number 1, 1999
University of Białystok

Properties of the Trigonometric Function

Takashi Mitsuishi
Shinshu University
Nagano

Yuguang Yang
Shinshu University
Nagano

Summary. This article introduces the monotone increasing and the monotone decreasing of *sinus* and *cosine*, and definitions of hyperbolic *sinus*, hyperbolic *cosine* and hyperbolic *tangent*, and some related formulas about them.

MML Identifier: SIN_COS2.

The papers [21], [6], [17], [22], [4], [14], [15], [20], [2], [19], [3], [18], [13], [5], [7], [8], [16], [9], [10], [1], [23], [11], and [12] provide the notation and terminology for this paper.

1. Monotone Increasing and Monotone Decreasing of Sinus and Cosine

We adopt the following rules: p, q, r, t_1 are elements of $\mathbb{R}$ and n is a natural number.

Next we state a number of propositions:

(1) If $p \geqslant 0$ and $r \geqslant 0$, then $p + r \geqslant 2 \cdot \sqrt{p \cdot r}$.

(2) sin is increasing on $]0, \frac{\mathrm{Pai}}{2}[$.

(3) sin is decreasing on $]\frac{\mathrm{Pai}}{2}, \mathrm{Pai}[$.

(4) cos is decreasing on $]0, \frac{\mathrm{Pai}}{2}[$.

(5) cos is decreasing on $]\frac{\mathrm{Pai}}{2}, \mathrm{Pai}[$.

(6) sin is decreasing on $]\mathrm{Pai}, \frac{3}{2} \cdot \mathrm{Pai}[$.

(7) sin is increasing on $]\frac{3}{2} \cdot \mathrm{Pai}, 2 \cdot \mathrm{Pai}[$.

(8) cos is increasing on $]\mathrm{Pai}, \frac{3}{2} \cdot \mathrm{Pai}[$.

ISSN 1426–2630

(9) cos is increasing on $]\frac{3}{2} \cdot \mathrm{Pai}, 2 \cdot \mathrm{Pai}[$.

(10) $(\sin)(t_1) = (\sin)(2 \cdot \mathrm{Pai} \cdot n + t_1)$.

(11) $(\cos)(t_1) = (\cos)(2 \cdot \mathrm{Pai} \cdot n + t_1)$.

2. Hyperbolic Sinus, Hyperbolic Cosine and Hyperbolic Tangent

The partial function sinh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 1) $\mathrm{dom}\,\mathrm{sinh} = \mathbb{R}$ and for every real number d holds $(\sinh)(d) = \frac{(\exp)(d)-(\exp)(-d)}{2}$.

Let d be a real number. The functor $\sinh d$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 2) $\sinh d = (\sinh)(d)$.

The partial function cosh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{cosh} = \mathbb{R}$ and for every real number d holds $(\cosh)(d) = \frac{(\exp)(d)+(\exp)(-d)}{2}$.

Let d be a real number. The functor $\cosh d$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 4) $\cosh d = (\cosh)(d)$.

The partial function tanh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 5) $\mathrm{dom}\,\mathrm{tanh} = \mathbb{R}$ and for every real number d holds $(\tanh)(d) = \frac{(\exp)(d)-(\exp)(-d)}{(\exp)(d)+(\exp)(-d)}$.

Let d be a real number. The functor $\tanh d$ yields an element of $\mathbb{R}$ and is defined as follows:

(Def. 6) $\tanh d = (\tanh)(d)$.

We now state a number of propositions:

(12) $(\exp)(p+q) = (\exp)(p) \cdot (\exp)(q)$.

(13) $(\exp)(0) = 1$.

(14) $(\cosh)(p)^{\mathbf{2}} - (\sinh)(p)^{\mathbf{2}} = 1$ and $(\cosh)(p) \cdot (\cosh)(p) - (\sinh)(p) \cdot (\sinh)(p) = 1$.

(15) $(\cosh)(p) \neq 0$ and $(\cosh)(p) > 0$ and $(\cosh)(0) = 1$.

(16) $(\sinh)(0) = 0$.

(17) $(\tanh)(p) = \frac{(\sinh)(p)}{(\cosh)(p)}$.

(18) $(\sinh)(p)^{\mathbf{2}} = \frac{1}{2} \cdot ((\cosh)(2 \cdot p) - 1)$ and $(\cosh)(p)^{\mathbf{2}} = \frac{1}{2} \cdot ((\cosh)(2 \cdot p) + 1)$.

(19) $(\cosh)(-p) = (\cosh)(p)$ and $(\sinh)(-p) = -(\sinh)(p)$ and $(\tanh)(-p) = -(\tanh)(p)$.

(20) $(\cosh)(p+r) = (\cosh)(p) \cdot (\cosh)(r) + (\sinh)(p) \cdot (\sinh)(r)$ and $(\cosh)(p-r) = (\cosh)(p) \cdot (\cosh)(r) - (\sinh)(p) \cdot (\sinh)(r)$.

(21) $(\sinh)(p+r) = (\sinh)(p)\cdot(\cosh)(r)+(\cosh)(p)\cdot(\sinh)(r)$ and $(\sinh)(p-r) = (\sinh)(p)\cdot(\cosh)(r)-(\cosh)(p)\cdot(\sinh)(r)$.

(22) $(\tanh)(p+r) = \frac{(\tanh)(p)+(\tanh)(r)}{1+(\tanh)(p)\cdot(\tanh)(r)}$ and $(\tanh)(p-r) = \frac{(\tanh)(p)-(\tanh)(r)}{1-(\tanh)(p)\cdot(\tanh)(r)}$.

(23) $(\sinh)(2\cdot p) = 2\cdot(\sinh)(p)\cdot(\cosh)(p)$ and $(\cosh)(2\cdot p) = 2\cdot(\cosh)(p)^{\mathbf{2}}-1$ and $(\tanh)(2\cdot p) = \frac{2\cdot(\tanh)(p)}{1+(\tanh)(p)^{\mathbf{2}}}$.

(24) $(\sinh)(p)^{\mathbf{2}}-(\sinh)(q)^{\mathbf{2}} = (\sinh)(p+q)\cdot(\sinh)(p-q)$ and $(\sinh)(p+q)\cdot(\sinh)(p-q) = (\cosh)(p)^{\mathbf{2}}-(\cosh)(q)^{\mathbf{2}}$ and $(\sinh)(p)^{\mathbf{2}}-(\sinh)(q)^{\mathbf{2}} = (\cosh)(p)^{\mathbf{2}}-(\cosh)(q)^{\mathbf{2}}$.

(25) $(\sinh)(p)^{\mathbf{2}}+(\cosh)(q)^{\mathbf{2}} = (\cosh)(p+q)\cdot(\cosh)(p-q)$ and $(\cosh)(p+q)\cdot(\cosh)(p-q) = (\cosh)(p)^{\mathbf{2}}+(\sinh)(q)^{\mathbf{2}}$ and $(\sinh)(p)^{\mathbf{2}}+(\cosh)(q)^{\mathbf{2}} = (\cosh)(p)^{\mathbf{2}}+(\sinh)(q)^{\mathbf{2}}$.

(26) $(\sinh)(p)+(\sinh)(r) = 2\cdot(\sinh)(\frac{p}{2}+\frac{r}{2})\cdot(\cosh)(\frac{p}{2}-\frac{r}{2})$ and $(\sinh)(p)-(\sinh)(r) = 2\cdot(\sinh)(\frac{p}{2}-\frac{r}{2})\cdot(\cosh)(\frac{p}{2}+\frac{r}{2})$.

(27) $(\cosh)(p)+(\cosh)(r) = 2\cdot(\cosh)(\frac{p}{2}+\frac{r}{2})\cdot(\cosh)(\frac{p}{2}-\frac{r}{2})$ and $(\cosh)(p)-(\cosh)(r) = 2\cdot(\sinh)(\frac{p}{2}-\frac{r}{2})\cdot(\sinh)(\frac{p}{2}+\frac{r}{2})$.

(28) $(\tanh)(p)+(\tanh)(r) = \frac{(\sinh)(p+r)}{(\cosh)(p)\cdot(\cosh)(r)}$ and $(\tanh)(p)-(\tanh)(r) = \frac{(\sinh)(p-r)}{(\cosh)(p)\cdot(\cosh)(r)}$.

(29) $((\cosh)(p)+(\sinh)(p))^{n}_{\mathbb{N}} = (\cosh)(n\cdot p)+(\sinh)(n\cdot p)$.

One can check the following observations:

* sinh is total,
* cosh is total, and
* tanh is total.

One can prove the following propositions:

(30) $\operatorname{dom}\sinh = \mathbb{R}$ and $\operatorname{dom}\cosh = \mathbb{R}$ and $\operatorname{dom}\tanh = \mathbb{R}$.

(31) sinh is differentiable in p and $(\sinh)'(p) = (\cosh)(p)$.

(32) cosh is differentiable in p and $(\cosh)'(p) = (\sinh)(p)$.

(33) tanh is differentiable in p and $(\tanh)'(p) = \frac{1}{(\cosh)(p)^{\mathbf{2}}}$.

(34) sinh is differentiable on $\mathbb{R}$ and $(\sinh)'(p) = (\cosh)(p)$.

(35) cosh is differentiable on $\mathbb{R}$ and $(\cosh)'(p) = (\sinh)(p)$.

(36) tanh is differentiable on $\mathbb{R}$ and $(\tanh)'(p) = \frac{1}{(\cosh)(p)^{\mathbf{2}}}$.

(37) $(\cosh)(p) \geqslant 1$.

(38) sinh is continuous in p.

(39) cosh is continuous in p.

(40) tanh is continuous in p.

(41) sinh is continuous on $\mathbb{R}$.

(42) cosh is continuous on $\mathbb{R}$.

(43) tanh is continuous on $\mathbb{R}$.

(44) $(\tanh)(p) < 1$ and $(\tanh)(p) > -1$.

References

[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Formalized Mathematics*, 4(**1**):121–124, 1993.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[7] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[8] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[9] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[10] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[13] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.
[14] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[15] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[16] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[19] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[23] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received March 13, 1999