FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

Technical Preliminaries to Algebraic Specifications

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MML Identifier: ALGSPEC1.

The papers [15], [3], [11], [5], [6], [7], [8], [4], [10], [13], [2], [9], [12], [1], [16], [17], and [14] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all functions f, g, h such that $\operatorname{dom} f \cap \operatorname{dom} g \subseteq \operatorname{dom} h$ holds $f{+\cdot}g{+\cdot}h = g{+\cdot}f{+\cdot}h$.

(2) For all functions f, g, h such that $f \subseteq g$ and $\operatorname{rng} h \cap \operatorname{dom} g \subseteq \operatorname{dom} f$ holds $g \cdot h = f \cdot h$.

(3) For all functions f, g, h such that $\operatorname{dom} f \subseteq \operatorname{rng} g$ and $\operatorname{dom} f$ misses $\operatorname{rng} h$ and $g^{\circ} \operatorname{dom} h$ misses $\operatorname{dom} f$ holds $f \cdot (g{+\cdot}h) = f \cdot g$.

(4) For all functions f_1, f_2, g_1, g_2 such that $f_1 \approx f_2$ and $g_1 \approx g_2$ holds $f_1 \cdot g_1 \approx f_2 \cdot g_2$.

(5) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \subseteq g$, then $f^* \subseteq g^*$.

(6) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \approx g$, then $f^* \approx g^*$.

Let X be a set and let f be a function. The functor X-indexing f yielding a many sorted set indexed by X is defined as follows:

(Def. 1) X-indexing $f = \mathrm{id}_X{+\cdot}f{\restriction}X$.

We now state a number of propositions:

ISSN 1426–2630

(7) For every set X and for every function f holds $\mathrm{rng}(X\text{-indexing}\, f) = (X \setminus \mathrm{dom}\, f) \cup f^{\circ}X$.

(8) For every non empty set X and for every function f and for every element x of X holds $(X\text{-indexing}\, f)(x) = (\mathrm{id}_X{+\cdot}f)(x)$.

(9) For all sets X, x and for every function f such that $x \in X$ holds if $x \in \mathrm{dom}\, f$, then $(X\text{-indexing}\, f)(x) = f(x)$ and if $x \notin \mathrm{dom}\, f$, then $(X\text{-indexing}\, f)(x) = x$.

(10) For every set X and for every function f such that $\mathrm{dom}\, f = X$ holds $X\text{-indexing}\, f = f$.

(11) For every set X and for every function f holds $X\text{-indexing}(X\text{-indexing}\, f) = X\text{-indexing}\, f$.

(12) For every set X and for every function f holds $X\text{-indexing}(\mathrm{id}_X{+\cdot}f) = X\text{-indexing}\, f$.

(13) For every set X and for every function f such that $f \subseteq \mathrm{id}_X$ holds $X\text{-indexing}\, f = \mathrm{id}_X$.

(14) For every set X holds $X\text{-indexing}\, \emptyset = \mathrm{id}_X$.

(15) For every set X and for every function f holds $X\text{-indexing}\, f{\restriction}X = X\text{-indexing}\, f$.

(16) For every set X and for every function f such that $X \subseteq \mathrm{dom}\, f$ holds $X\text{-indexing}\, f = f{\restriction}X$.

(17) For every function f holds $\emptyset\text{-indexing}\, f = \emptyset$.

(18) For all sets X, Y and for every function f such that $X \subseteq Y$ holds $(Y\text{-indexing}\, f){\restriction}X = X\text{-indexing}\, f$.

(19) For all sets X, Y and for every function f holds $(X \cup Y)\text{-indexing}\, f = (X\text{-indexing}\, f){+\cdot}(Y\text{-indexing}\, f)$.

(20) For all sets X, Y and for every function f holds $X\text{-indexing}\, f \approx Y\text{-indexing}\, f$.

(21) For all sets X, Y and for every function f holds $(X \cup Y)\text{-indexing}\, f = (X\text{-indexing}\, f) \cup (Y\text{-indexing}\, f)$.

(22) For every non empty set X and for all functions f, g such that $\mathrm{rng}\, g \subseteq X$ holds $(X\text{-indexing}\, f) \cdot g = (\mathrm{id}_X{+\cdot}f) \cdot g$.

(23) For all functions f, g such that $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$ and $\mathrm{rng}\, g$ misses $\mathrm{dom}\, f$ and for every set X holds $f \cdot (X\text{-indexing}\, g) = f{\restriction}X$.

Let f be a function. A function is called a rng-retraction of f if:

(Def. 2) $\mathrm{dom}\,\mathrm{it} = \mathrm{rng}\, f$ and $f \cdot \mathrm{it} = \mathrm{id}_{\mathrm{rng}\, f}$.

We now state several propositions:

(24) For every function f and for every rng-retraction g of f holds $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$.

(25) Let f be a function, g be a rng-retraction of f, and x be a set. If $x \in \operatorname{rng} f$, then $g(x) \in \operatorname{dom} f$ and $f(g(x)) = x$.

(26) For every function f such that f is one-to-one holds f^{-1} is a rng-retraction of f.

(27) For every function f such that f is one-to-one and for every rng-retraction g of f holds $g = f^{-1}$.

(28) Let f_1, f_2 be functions. Suppose $f_1 \approx f_2$. Let g_1 be a rng-retraction of f_1 and g_2 be a rng-retraction of f_2. Then $g_1{+\cdot}g_2$ is a rng-retraction of $f_1{+\cdot}f_2$.

(29) Let f_1, f_2 be functions. Suppose $f_1 \subseteq f_2$. Let g_1 be a rng-retraction of f_1. Then there exists a rng-retraction g_2 of f_2 such that $g_1 \subseteq g_2$.

2. Replacement in Signature

Let S be a non empty non void many sorted signature and let f, g be functions. We say that f and g form a replacement in S if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let o_1, o_2 be operation symbols of S. Suppose $(\mathrm{id}_{\text{the operation symbols of } S}{+\cdot}g)(o_1) = (\mathrm{id}_{\text{the operation symbols of } S}{+\cdot}g)(o_2)$. Then

(i) $(\mathrm{id}_{\text{the carrier of } S}{+\cdot}f) \cdot \mathrm{Arity}(o_1) = (\mathrm{id}_{\text{the carrier of } S}{+\cdot}f) \cdot \mathrm{Arity}(o_2)$, and

(ii) $(\mathrm{id}_{\text{the carrier of } S}{+\cdot}f)$(the result sort of o_1) $= (\mathrm{id}_{\text{the carrier of } S}{+\cdot}f)$(the result sort of o_2).

One can prove the following propositions:

(30) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if for all operation symbols o_1, o_2 of S such that ((the operation symbols of S)-indexing g)(o_1) = ((the operation symbols of S)-indexing g)(o_2) holds ((the carrier of S)-indexing f)$\cdot\mathrm{Arity}(o_1)$ = ((the carrier of S)-indexing f)$\cdot$ $\mathrm{Arity}(o_2)$ and ((the carrier of S)-indexing f)(the result sort of o_1) = ((the carrier of S)-indexing f)(the result sort of o_2).

(31) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if (the carrier of S)-indexing f and (the operation symbols of S)-indexing g form a replacement in S.

In the sequel S, S' denote non void signatures and f, g denote functions.

One can prove the following four propositions:

(32) If f and g form morphism between S and S', then f and g form a replacement in S.

(33) f and $\emptyset$ form a replacement in S.

(34) If g is one-to-one and (the operation symbols of S) $\cap \operatorname{rng} g \subseteq \operatorname{dom} g$, then f and g form a replacement in S.

(35) If g is one-to-one and $\operatorname{rng} g$ misses the operation symbols of S, then f and g form a replacement in S.

Let X be a set, let Y be a non empty set, let a be a function from Y into X^*, and let r be a function from Y into X. Observe that $\langle X, Y, a, r \rangle$ is non void.

Let S be a non empty non void many sorted signature and let f, g be functions. Let us assume that f and g form a replacement in S. The functor S with-replacement(f, g) yields a strict non empty non void many sorted signature and is defined by the conditions (Def. 4).

(Def. 4)(i) (The carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g),

(ii) the carrier of S with-replacement(f, g) = rng((the carrier of S) -indexing f), and

(iii) the operation symbols of S with-replacement(f, g) = rng((the operation symbols of S) -indexing g).

The following propositions are true:

(36) Let S_1, S_2 be non void signatures, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Then $f^* \cdot$ the arity of S_1 = (the arity of S_2) $\cdot\, g$.

(37) Suppose f and g form a replacement in S. Then (the carrier of S) -indexing f is a function from the carrier of S into the carrier of S with-replacement(f, g).

(38) Suppose f and g form a replacement in S. Let f' be a function from the carrier of S into the carrier of S with-replacement(f, g). Suppose f' = (the carrier of S) -indexing f. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the arity of S with-replacement(f, g) = $f'^* \cdot$ the arity of $S \cdot g'$.

(39) Suppose f and g form a replacement in S. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the result sort of S with-replacement(f, g) = ((the carrier of S) -indexing f) $\cdot$ the result sort of $S \cdot g'$.

(40) If f and g form morphism between S and S', then S with-replacement(f, g) is a subsignature of S'.

(41) f and g form a replacement in S if and only if (the carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g).

(42) Suppose $\operatorname{dom} f \subseteq$ the carrier of S and $\operatorname{dom} g \subseteq$ the operation symbols of S and f and g form a replacement in S. Then $\mathrm{id}_{\text{the carrier of } S} {+\cdot} f$ and $\mathrm{id}_{\text{the operation symbols of } S} {+\cdot} g$ form morphism be-

tween S and S with-replacement(f, g).

(43) Suppose $\mathrm{dom}\, f =$ the carrier of S and $\mathrm{dom}\, g =$ the operation symbols of S and f and g form a replacement in S. Then f and g form morphism between S and S with-replacement(f, g).

(44) If f and g form a replacement in S, then S with-replacement((the carrier of S) -indexing $f, g) = S$ with-replacement(f, g).

(45) If f and g form a replacement in S, then S with-replacement$(f,$ (the operation symbols of S) -indexing $g) = S$ with-replacement(f, g).

3. Signature Extensions

Let S be a signature. A signature is called an extension of S if:

(Def. 5) S is a subsignature of it.

The following propositions are true:

(46) For all signatures S, E holds S is a subsignature of E iff E is an extension of S.

(47) Every signature S is an extension of S.

(48) For every signature S_1 and for every extension S_2 of S_1 holds every extension of S_2 is an extension of S_1.

(49) For all non empty signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 {+\cdot} S_2$ is an extension of S_1.

(50) For all non empty signatures S_1, S_2 holds $S_1 {+\cdot} S_2$ is an extension of S_2.

(51) Let S_1, S_2, S be non empty many sorted signatures and f_1, g_1, f_2, g_2 be functions. Suppose $f_1 \approx f_2$ and f_1 and g_1 form morphism between S_1 and S and f_2 and g_2 form morphism between S_2 and S. Then $f_1 {+\cdot} f_2$ and $g_1 {+\cdot} g_2$ form morphism between $S_1 {+\cdot} S_2$ and S.

(52) Let S_1, S_2, E be non empty signatures. Then E is an extension of S_1 and an extension of S_2 if and only if $S_1 \approx S_2$ and E is an extension of $S_1 {+\cdot} S_2$.

Let S be a non empty signature. One can check that every extension of S is non empty.

Let S be a non void signature. One can verify that every extension of S is non void.

One can prove the following proposition

(53) For all signatures S, T such that S is empty holds T is an extension of S.

Let S be a signature. One can check that there exists an extension of S which is non empty, non void, and strict.

The following three propositions are true:

(54) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then f and g form a replacement in S.

(55) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then E with-replacement(f, g) is an extension of S with-replacement(f, g).

(56) Let S_1, S_2 be non void signatures. Suppose $S_1 \approx S_2$. Let f, g be functions. If f and g form a replacement in $S_1{+}{\cdot}S_2$, then $(S_1{+}{\cdot}S_2)$ with-replacement$(f, g) =$ $(S_1$ with-replacement$(f, g)){+}{\cdot}(S_2$ with-replacement$(f, g))$.

4. Algebras

Algebra is defined by:

(Def. 6) There exists a non void signature S such that it is a feasible algebra over S.

Let S be a signature. An algebra is called an algebra of S if:

(Def. 7) There exists a non void extension E of S such that it is a feasible algebra over E.

One can prove the following propositions:

(57) For every non void signature S holds every feasible algebra over S is an algebra of S.

(58) For every signature S and for every extension E of S holds every algebra of E is an algebra of S.

(59) Let S be a signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Then the carrier of $S \subseteq$ the carrier of E and the operation symbols of $S \subseteq$ the operation symbols of E.

(60) Let S be a non void signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Let o be an operation symbol of S. Then (the characteristics of A)(o) is a function from (the sorts of A)$^{\#}$(Arity(o)) into (the sorts of A)(the result sort of o).

(61) Let S be a non empty signature, A be an algebra of S, and E be a non empty many sorted signature. If A is an algebra over E, then A is an algebra over $E{+}{\cdot}S$.

(62) Let S_1, S_2 be non empty signatures and A be an algebra over S_1. Suppose A is an algebra over S_2. Then the carrier of $S_1 =$ the carrier of S_2 and the operation symbols of $S_1 =$ the operation symbols of S_2.

(63) For every non void signature S and for every non-empty disjoint algebra A over S holds the sorts of A are one-to-one.

(64) Let S be a non void signature, A be a disjoint algebra over S, and C_1, C_2 be components of the sorts of A. Then $C_1 = C_2$ or C_1 misses C_2.

(65) Let S, S' be non void signatures and A be a non-empty disjoint algebra over S. Suppose A is an algebra over S'. Then the many sorted signature of $S =$ the many sorted signature of S'.

(66) Let S' be a non void signature and A be a non-empty disjoint algebra over S. If A is an algebra of S', then S is an extension of S'.

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Received September 7, 1999