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Scott-Continuous Functions. Part II[1]

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The terminology and notation used here are introduced in the following articles: [13], [5], [1], [16], [6], [14], [11], [18], [17], [12], [15], [7], [3], [4], [10], [2], [8], [19], and [9].

1. Preliminaries

One can prove the following proposition

(1) Let S, T be up-complete Scott top-lattices and M be a subset of $\mathrm{SCMaps}(S,T)$. Then $\bigsqcup_{\mathrm{SCMaps}(S,T)} M$ is a continuous map from S into T.

Let S be a non empty relational structure and let T be a non empty reflexive relational structure. One can check that every map from S into T which is constant is also monotone.

Let S be a non empty relational structure, let T be a reflexive non empty relational structure, and let a be an element of the carrier of T. One can check that $S \longmapsto a$ is monotone.

One can prove the following propositions:

(2) Let S be a non empty relational structure and T be a lower-bounded antisymmetric reflexive non empty relational structure. Then $\perp_{\mathrm{MonMaps}(S,T)} = S \longmapsto \perp_T$.

(3) Let S be a non empty relational structure and T be an upper-bounded antisymmetric reflexive non empty relational structure. Then $\top_{\mathrm{MonMaps}(S,T)} = S \longmapsto \top_T$.

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(4) Let S, T be complete lattices, f be a monotone map from S into T, and x be an element of S. Then $f(x) = \sup(f^{\circ}{\downarrow}x)$.

(5) Let S, T be complete lower-bounded lattices, f be a monotone map from S into T, and x be an element of S. Then $f(x) = \bigsqcup_T\{f(w); w$ ranges over elements of S: $w \leqslant x\}$.

(6) Let S be a relational structure, T be a non empty relational structure, and F be a subset of $T^{\text{the carrier of } S}$. Then $\sup F$ is a map from S into T.

2. On the Scott Continuity of Maps

Let X_1, X_2, Y be non empty relational structures, let f be a map from $[: X_1, X_2 :]$ into Y, and let x be an element of the carrier of X_1. The functor $\mathrm{Proj}(f, x)$ yields a map from X_2 into Y and is defined as follows:

(Def. 1) $\mathrm{Proj}(f, x) = (\mathrm{curry}\, f)(x)$.

For simplicity, we use the following convention: X_1, X_2, Y denote non empty relational structures, f denotes a map from $[: X_1, X_2 :]$ into Y, x denotes an element of the carrier of X_1, and y denotes an element of the carrier of X_2.

We now state the proposition

(7) For every element y of the carrier of X_2 holds $(\mathrm{Proj}(f, x))(y) = f(\langle x, y\rangle)$.

Let X_1, X_2, Y be non empty relational structures, let f be a map from $[: X_1, X_2 :]$ into Y, and let y be an element of the carrier of X_2. The functor $\mathrm{Proj}(f, y)$ yielding a map from X_1 into Y is defined by:

(Def. 2) $\mathrm{Proj}(f, y) = (\mathrm{curry}'\, f)(y)$.

The following propositions are true:

(8) For every element x of the carrier of X_1 holds $(\mathrm{Proj}(f, y))(x) = f(\langle x, y\rangle)$.

(9) Let R, S, T be non empty relational structures, f be a map from $[: R, S :]$ into T, a be an element of R, and b be an element of S. Then $(\mathrm{Proj}(f, a))(b) = (\mathrm{Proj}(f, b))(a)$.

Let S be a non empty relational structure and let T be a non empty reflexive relational structure. Observe that there exists a map from S into T which is antitone.

The following two propositions are true:

(10) Let R, S, T be non empty reflexive relational structures, f be a map from $[: R, S :]$ into T, a be an element of the carrier of R, and b be an element of the carrier of S. If f is monotone, then $\mathrm{Proj}(f, a)$ is monotone and $\mathrm{Proj}(f, b)$ is monotone.

(11) Let R, S, T be non empty reflexive relational structures, f be a map from $[: R, S :]$ into T, a be an element of the carrier of R, and b be an element of the carrier of S. If f is antitone, then $\mathrm{Proj}(f, a)$ is antitone and $\mathrm{Proj}(f, b)$ is antitone.

Let R, S, T be non empty reflexive relational structures, let f be a monotone map from $[: R, S :]$ into T, and let a be an element of the carrier of R. Note that $\mathrm{Proj}(f, a)$ is monotone.

Let R, S, T be non empty reflexive relational structures, let f be a monotone map from $[: R, S :]$ into T, and let b be an element of the carrier of S. Note that $\mathrm{Proj}(f, b)$ is monotone.

Let R, S, T be non empty reflexive relational structures, let f be an antitone map from $[: R, S :]$ into T, and let a be an element of the carrier of R. Observe that $\mathrm{Proj}(f, a)$ is antitone.

Let R, S, T be non empty reflexive relational structures, let f be an antitone map from $[: R, S :]$ into T, and let b be an element of the carrier of S. Note that $\mathrm{Proj}(f, b)$ is antitone.

We now state several propositions:

(12) Let R, S, T be lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is monotone and $\mathrm{Proj}(f, b)$ is monotone. Then f is monotone.

(13) Let R, S, T be lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is antitone and $\mathrm{Proj}(f, b)$ is antitone. Then f is antitone.

(14) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, b be an element of S, and X be a subset of R. Then $(\mathrm{Proj}(f, b))^\circ X = f^\circ [: X, \{b\} :]$.

(15) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, b be an element of R, and X be a subset of S. Then $(\mathrm{Proj}(f, b))^\circ X = f^\circ [: \{b\}, X :]$.

(16) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, a be an element of R, and b be an element of S. Suppose f is directed-sups-preserving. Then $\mathrm{Proj}(f, a)$ is directed-sups-preserving and $\mathrm{Proj}(f, b)$ is directed-sups-preserving.

(17) Let R, S, T be lattices, f be a monotone map from $[: R, S :]$ into T, a be an element of R, b be an element of S, and X be a directed subset of $[: R, S :]$. If $\sup f^\circ X$ exists in T and $a \in \pi_1(X)$ and $b \in \pi_2(X)$, then $f(\langle a, b \rangle) \leqslant \sup(f^\circ X)$.

(18) Let R, S, T be complete lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is directed-sups-preserving and $\mathrm{Proj}(f, b)$ is directed-sups-preserving. Then f is directed-sups-preserving.

(19) Let S be a non empty 1-sorted structure, T be a non empty relational

structure, and f be a set. Then f is an element of $T^{\text{the carrier of } S}$ if and only if f is a map from S into T.

3. The Poset of Continuous Maps

Let S be a topological structure and let T be a non empty FR-structure. The functor $[S \to T]$ yielding a strict relational structure is defined by the conditions (Def. 3).

(Def. 3)(i) $[S \to T]$ is a full relational substructure of $T^{\text{the carrier of } S}$, and

(ii) for every set x holds $x \in$ the carrier of $([S \to T])$ iff there exists a map f from S into T such that $x = f$ and f is continuous.

Let S be a non empty topological space and let T be a non empty topological space-like FR-structure. Observe that $[S \to T]$ is non empty.

Let S be a non empty topological space and let T be a non empty topological space-like FR-structure. Note that $[S \to T]$ is constituted functions.

One can prove the following propositions:

(20) Let S be a non empty topological space, T be a non empty reflexive topological space-like FR-structure, and x, y be elements of $[S \to T]$. Then $x \leqslant y$ if and only if for every element i of S holds $\langle x(i), y(i) \rangle \in$ the internal relation of T.

(21) Let S be a non empty topological space, T be a non empty reflexive topological space-like FR-structure, and x be a set. Then x is a continuous map from S into T if and only if x is an element of $[S \to T]$.

Let S be a non empty topological space and let T be a non empty reflexive topological space-like FR-structure. Note that $[S \to T]$ is reflexive.

Let S be a non empty topological space and let T be a non empty transitive topological space-like FR-structure. Note that $[S \to T]$ is transitive.

Let S be a non empty topological space and let T be a non empty antisymmetric topological space-like FR-structure. One can check that $[S \to T]$ is antisymmetric.

Let S be a non empty 1-sorted structure and let T be a non empty topological space-like FR-structure. One can verify that $T^{\text{the carrier of } S}$ is constituted functions.

One can prove the following three propositions:

(22) Let S be a non empty 1-sorted structure, T be a complete lattice, f, g, h be maps from S into T, and i be an element of S. If $h = \bigsqcup_{(T^{\text{the carrier of } S})}\{f, g\}$, then $h(i) = \sup\{f(i), g(i)\}$.

(23) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for

every element i of I holds $J(i)$ is a complete lattice. Let X be a subset of $\prod J$ and i be an element of I. Then $(\inf X)(i) = \inf \pi_i X$.

(24) Let S be a non empty 1-sorted structure, T be a complete lattice, f, g, h be maps from S into T, and i be an element of S. If $h = \bigsqcap_{(T^{\text{the carrier of } S})}\{f, g\}$, then $h(i) = \inf\{f(i), g(i)\}$.

Let S be a non empty 1-sorted structure and let T be a lattice. Observe that every element of $T^{\text{the carrier of } S}$ is function-like and relation-like.

Let S, T be top-lattices. One can check that every element of $[S \to T]$ is function-like and relation-like.

One can prove the following propositions:

(25) Let S be a non empty relational structure, T be a complete lattice, F be a non empty subset of $T^{\text{the carrier of } S}$, and i be an element of the carrier of S. Then $(\sup F)(i) = \bigsqcup_T \{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}$.

(26) Let S, T be complete top-lattices, F be a non empty subset of $[S \to T]$, and i be an element of the carrier of S. Then $(\bigsqcup_{(T^{\text{the carrier of } S})} F)(i) = \bigsqcup_T \{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}$.

In the sequel S denotes a non empty relational structure, T denotes a complete lattice, and i denotes an element of S.

Next we state two propositions:

(27) Let F be a non empty subset of $T^{\text{the carrier of } S}$ and D be a non empty subset of S. Then $(\sup F)^\circ D = \{\bigsqcup_T \{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}; i$ ranges over elements of S: $i \in D\}$.

(28) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a non empty subset of S. Then $(\bigsqcup_{(T^{\text{the carrier of } S})} F)^\circ D = \{\bigsqcup_T \{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}; i$ ranges over elements of S: $i \in D\}$.

The scheme *FraenkelF'RSS* deals with a non empty relational structure $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the following condition is met:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The following propositions are true:

(29) Let S, T be complete Scott top-lattices and F be a non empty subset of $[S \to T]$. Then $\bigsqcup_{(T^{\text{the carrier of } S})} F$ is a monotone map from S into T.

(30) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a directed non empty subset of S. Then $\bigsqcup_T \{\bigsqcup_T \{g(i); i$ ranges over elements of S: $i \in D\}; g$ ranges over elements of $T^{\text{the carrier of } S}$:

$g \in F\} = \bigsqcup_T \{\bigsqcup_T \{g'(i'); g'$ ranges over elements of $T^{\text{the carrier of } S}$: $g' \in F\}; i'$ ranges over elements of S: $i' \in D\}$.

(31) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a directed non empty subset of S. Then $\bigsqcup_T((\bigsqcup_{(T^{\text{the carrier of } S})} F)^{\circ} D) = (\bigsqcup_{(T^{\text{the carrier of } S})} F)(\sup D)$.

(32) Let S, T be complete Scott top-lattices and F be a non empty subset of $[S \to T]$. Then $\bigsqcup_{(T^{\text{the carrier of } S})} F \in$ the carrier of $([S \to T])$.

(33) Let S be a non empty relational structure and T be a lower-bounded antisymmetric non empty relational structure. Then $\bot_{T^{\text{the carrier of } S}} = S \longmapsto \bot_T$.

(34) Let S be a non empty relational structure and T be an upper-bounded antisymmetric non empty relational structure. Then $\top_{T^{\text{the carrier of } S}} = S \longmapsto \top_T$.

Let S be a non empty reflexive relational structure, let T be a complete lattice, and let a be an element of T. Note that $S \longmapsto a$ is directed-sups-preserving.

One can prove the following proposition

(35) Let S, T be complete Scott top-lattices. Then $[S \to T]$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

Let S, T be complete Scott top-lattices. Observe that $[S \to T]$ is complete.

We now state three propositions:

(36) For all non empty Scott complete top-lattices S, T holds $\bot_{[S \to T]} = S \longmapsto \bot_T$.

(37) For all non empty Scott complete top-lattices S, T holds $\top_{[S \to T]} = S \longmapsto \top_T$.

(38) For all Scott complete top-lattices S, T holds $\mathrm{SCMaps}(S, T) = [S \to T]$.

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