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Continuous Lattices between $\mathbf{T}_0$ Spaces[1]

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Summary. Formalization of [17, pp. 128–130], chapter II, section 4 (4.1 – 4.9).

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The terminology and notation used in this paper have been introduced in the following articles: [29], [16], [12], [13], [11], [1], [2], [32], [18], [30], [24], [25], [26], [27], [3], [9], [34], [35], [33], [28], [15], [21], [37], [10], [31], [20], [23], [5], [14], [6], [22], [8], [4], [19], [36], and [7].

Let I be a set and let J be a relational structure yielding many sorted set indexed by I. We introduce $I\text{-prod}_{\rm POS}\, J$ as a synonym of $\prod J$.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. One can check that $I\text{-prod}_{\rm POS}\, J$ is constituted functions.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. We introduce $I\text{-prod}_{\rm TOP}\, J$ as a synonym of $\prod J$.

Let X, Y be non empty topological spaces. The functor $[X \to Y]$ yields a non empty strict relational structure and is defined as follows:

(Def. 1) $[X \to Y] = [X \to \Omega Y]$.

Let X, Y be non empty topological spaces. Observe that $[X \to Y]$ is reflexive transitive and constituted functions.

Let X be a non empty topological space and let Y be a non empty T_0 topological space. Observe that $[X \to Y]$ is antisymmetric.

We now state three propositions:

(1) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into ΩY.

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(2) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into Y.

(3) Let X, Y be non empty topological spaces, a, b be elements of $[X \to Y]$, and f, g be maps from X into ΩY. If $a = f$ and $b = g$, then $a \leqslant b$ iff $f \leqslant g$.

Let X, Y be non empty topological spaces, let x be a point of X, and let A be a subset of the carrier of $([X \to Y])$. Then $\pi_x A$ is a subset of ΩY.

Let X, Y be non empty topological spaces, let x be a set, and let A be a non empty subset of the carrier of $([X \to Y])$. Observe that $\pi_x A$ is non empty.

We now state three propositions:

(4) Ω(the Sierpiński space) is a topological augmentation of $2^1_\subseteq$.

(5) Let X be a non empty topological space. Then there exists a map f from $\langle$the topology of X, $\subseteq\rangle$ into $[X \to$ the Sierpiński space$]$ such that f is isomorphic and for every open subset V of X holds $f(V) = \chi_{V,\text{the carrier of } X}$.

(6) Let X be a non empty topological space. Then $\langle$the topology of X, $\subseteq\rangle$ and $[X \to$ the Sierpiński space$]$ are isomorphic.

Let X, Y, Z be non empty topological spaces and let f be a continuous map from Y into Z. The functor $[X \to f]$ yields a map from $[X \to Y]$ into $[X \to Z]$ and is defined by:

(Def. 2) For every continuous map g from X into Y holds $([X \to f])(g) = f \cdot g$.

The functor $[f \to X]$ yields a map from $[Z \to X]$ into $[Y \to X]$ and is defined by:

(Def. 3) For every continuous map g from Z into X holds $([f \to X])(g) = g \cdot f$.

The following propositions are true:

(7) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \to Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

(8) For every non empty topological space X and for every monotone convergence T_0-space Y holds $[X \to Y]$ is up-complete.

(9) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[X \to f]$ is monotone.

(10) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[X \to f]$ is idempotent.

(11) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[f \to X]$ is monotone.

(12) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[f \to X]$ is idempotent.

(13) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of X, and A be a subset of $[X \to Y]$.

Then $\pi_x([X \to f])^\circ A = f^\circ \pi_x A$.

(14) Let X be a non empty topological space, Y, Z be monotone convergence T_0-spaces, and f be a continuous map from Y into Z. Then $[X \to f]$ is directed-sups-preserving.

(15) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of Y, and A be a subset of $[Z \to X]$. Then $\pi_x([f \to X])^\circ A = \pi_{f(x)} A$.

(16) Let Y, Z be non empty topological spaces, X be a monotone convergence T_0-space, and f be a continuous map from Y into Z. Then $[f \to X]$ is directed-sups-preserving.

(17) Let X, Z be non empty topological spaces and Y be a non empty subspace of Z. Then $[X \to Y]$ is a full relational substructure of $[X \to Z]$.

(18) Let Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. Suppose f is a retraction. Then ΩY is a directed-sups-inheriting relational substructure of ΩZ.

(19) Let X be a non empty topological space, Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. If f is a retraction, then $[X \to f]$ is a retraction of $[X \to Z]$ into $[X \to Y]$.

(20) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. If Y is a retract of Z, then $[X \to Y]$ is a retract of $[X \to Z]$.

(21) Let X, Y, Z be non empty topological spaces and f be a continuous map from Y into Z. If f is a homeomorphism, then $[X \to f]$ is isomorphic.

(22) Let X, Y, Z be non empty topological spaces. If Y and Z are homeomorphic, then $[X \to Y]$ and $[X \to Z]$ are isomorphic.

(23) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. Suppose Y is a retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(24) Let X be a non empty topological space and Y, Z be monotone convergence T_0-spaces. Suppose Y is a topological retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(25) Let Y be a non trivial T_0-space. Suppose Y is not a T_1 space. Then the Sierpiński space is a topological retract of Y.

(26) Let X be a non empty topological space and Y be a non trivial T_0-space. If $[X \to Y]$ has l.u.b.'s, then Y is not a T_1 space.

One can check that the Sierpiński space is non trivial and monotone convergence.

One can verify that there exists a T_0-space which is injective, monotone convergence, and non trivial.

The following propositions are true:

(27) Let X be a non empty topological space and Y be a monotone convergence non trivial T_0-space. If $[X \to Y]$ is complete and continuous, then $\langle$the topology of X, $\subseteq\rangle$ is continuous.

(28) Let X be a non empty topological space, x be a point of X, and Y be a monotone convergence T_0-space. Then there exists a directed-sups-preserving projection map F from $[X \to Y]$ into $[X \to Y]$ such that

(i) for every continuous map f from X into Y holds $F(f) = X \longmapsto f(x)$, and

(ii) there exists a continuous map h from X into X such that $h = X \longmapsto x$ and $F = [h \to Y]$.

(29) Let X be a non empty topological space and Y be a monotone convergence T_0-space. If $[X \to Y]$ is complete and continuous, then ΩY is complete and continuous.

(30) Let X be a non empty 1-sorted structure, I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, f be a map from X into I-prod$_{\rm TOP}\, J$, and i be an element of I. Then $(\mathrm{commute}(f))(i) = \mathrm{proj}(J, i) \cdot f$.

(31) For every 1-sorted structure S and for every set M holds the support of $M \longmapsto S = M \longmapsto$ the carrier of S.

(32) Let X, Y be non empty topological spaces, M be a non empty set, and f be a continuous map from X into M-prod$_{\rm TOP}(M \longmapsto Y)$. Then commute($f$) is a function from M into the carrier of $([X \to Y])$.

(33) For all non empty topological spaces X, Y holds the carrier of $([X \to Y]) \subseteq$ (the carrier of Y)$^{\text{the carrier of } X}$.

(34) Let X, Y be non empty topological spaces, M be a non empty set, and f be a function from M into the carrier of $([X \to Y])$. Then commute(f) is a continuous map from X into M-prod$_{\rm TOP}(M \longmapsto Y)$.

(35) Let X be a non empty topological space and M be a non empty set. Then there exists a map F from $[X \to M$-prod$_{\rm TOP}(M \longmapsto$ the Sierpiński space)$]$ into M-prod$_{\rm POS}(M \longmapsto ([X \to$ the Sierpiński space$]))$ such that F is isomorphic and for every continuous map f from X into M-prod$_{\rm TOP}(M \longmapsto$ the Sierpiński space) holds $F(f) = \mathrm{commute}(f)$.

(36) Let X be a non empty topological space and M be a non empty set. Then $[X \to M$-prod$_{\rm TOP}(M \longmapsto$ the Sierpiński space)$]$ and M-prod$_{\rm POS}(M \longmapsto ([X \to$ the Sierpiński space$]))$ are isomorphic.

(37) Let X be a non empty topological space. Suppose $\langle$the topology of X, $\subseteq\rangle$ is continuous. Let Y be an injective T_0-space. Then $[X \to Y]$ is complete

and continuous.

Let us observe that there exists a top-lattice which is non trivial, complete, and Scott.

Next we state the proposition

(38) Let X be a non empty topological space and L be a non trivial complete Scott top-lattice. Then $[X \to L]$ is complete and continuous if and only if $\langle$the topology of X, $\subseteq\rangle$ is continuous and L is continuous.

Let f be a function. Observe that Union disjoint f is relation-like.

Let f be a function. The functor G_f yields a binary relation and is defined as follows:

(Def. 4) $G_f = (\mathrm{Union}\,\mathrm{disjoint}\, f)^{\smile}$.

In the sequel x, y are sets and f is a function.

We now state three propositions:

(39) $\langle x, y\rangle \in G_f$ iff $x \in \mathrm{dom}\, f$ and $y \in f(x)$.

(40) For every finite set X holds $\pi_1(X)$ is finite and $\pi_2(X)$ is finite.

(41) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and f be a map from X into S. If G_f is an open subset of $[\!: X, Y :\!]$, then f is continuous.

Let W be a binary relation and let X be a set. The functor $\Theta_X(W)$ yielding a function is defined by:

(Def. 5) $\mathrm{dom}\,\Theta_X(W) = X$ and for every x such that $x \in X$ holds $(\Theta_X(W))(x) = W^\circ\{x\}$.

One can prove the following proposition

(42) For every binary relation W and for every set X such that $\mathrm{dom}\, W \subseteq X$ holds $G_{\Theta_X(W)} = W$.

Let X, Y be topological spaces. Observe that every subset of the carrier of $[\!: X, Y :\!]$ is relation-like and every element of the topology of $[\!: X, Y :\!]$ is relation-like.

Next we state four propositions:

(43) Let X, Y be non empty topological spaces, W be an open subset of $[\!: X, Y :\!]$, and x be a point of X. Then $W^\circ\{x\}$ is an open subset of Y.

(44) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W be an open subset of $[\!: X, Y :\!]$. Then $\Theta_{\text{the carrier of } X}(W)$ is a continuous map from X into S.

(45) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W_1, W_2 be open subsets of $[\!: X, Y :\!]$. Suppose $W_1 \subseteq W_2$. Let f_1, f_2 be elements of $[X \to S]$. If $f_1 = \Theta_{\text{the carrier of } X}(W_1)$ and $f_2 = \Theta_{\text{the carrier of } X}(W_2)$, then $f_1 \leqslant f_2$.

(46) Let X, Y be non empty topological spaces and S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$. Then there exists a map F from $\langle$the topology of $[\!: X, Y :\!]$, $\subseteq\rangle$ into $[X \to S]$ such that F is monotone and for every open subset W of $[\!: X, Y :\!]$ holds $F(W) = \Theta_{\text{the carrier of } X}(W)$.

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