FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

Components and Basis of Topological Spaces[1]

Robert Milewski
University in Białystok

Summary. This article contains many facts about components and basis of topological spaces.

MML Identifier: `YELLOW15`.

The notation and terminology used here are introduced in the following papers: [21], [15], [1], [14], [6], [7], [19], [9], [8], [17], [2], [22], [18], [13], [12], [20], [16], [23], [11], [4], [5], [10], and [3].

1. Preliminaries

The scheme *SeqLambda1C* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and and states that:

> There exists a finite sequence p of elements of $\mathcal{B}$ such that $\operatorname{len} p = \mathcal{A}$ and for every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $p(i) = \mathcal{F}(i)$ and if not $\mathcal{P}[i]$, then $p(i) = \mathcal{G}(i)$

provided the following requirement is met:

- For every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $\mathcal{F}(i) \in \mathcal{B}$ and if not $\mathcal{P}[i]$, then $\mathcal{G}(i) \in \mathcal{B}$.

Let X be a set and let p be a finite sequence of elements of 2^X. Then $\operatorname{rng} p$ is a family of subsets of X.

Let us observe that *Boolean* is finite.

We now state two propositions:

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(2)[2] For every natural number i and for every finite set D holds D^i is finite.

(3) For every finite set T holds every family of subsets of T is finite.

Let T be a finite set. One can check that every family of subsets of T is finite.

Let T be a finite 1-sorted structure. One can verify that every family of subsets of T is finite.

One can prove the following proposition

(4) For every infinite set X there exist sets x, y such that $x \in X$ and $y \in X$ and $x \neq y$.

2. Components

Let X be a set, let p be a finite sequence of elements of 2^X, and let q be a finite sequence of elements of *Boolean*. The functor MergeSequence(p, q) yielding a finite sequence of elements of 2^X is defined as follows:

(Def. 1) $\operatorname{len} \operatorname{MergeSequence}(p, q) = \operatorname{len} p$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $(\operatorname{MergeSequence}(p, q))(i) = (q(i) = \mathit{true} \rightarrow p(i), X \setminus p(i))$.

One can prove the following propositions:

(5) Let X be a set, p be a finite sequence of elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then $\operatorname{dom} \operatorname{MergeSequence}(p, q) = \operatorname{dom} p$.

(6) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $q(i) = \mathit{true}$, then $(\operatorname{MergeSequence}(p, q))(i) = p(i)$.

(7) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $i \in \operatorname{dom} p$ and $q(i) = \mathit{false}$, then $(\operatorname{MergeSequence}(p, q))(i) = X \setminus p(i)$.

(8) For every set X and for every finite sequence q of elements of *Boolean* holds $\operatorname{len} \operatorname{MergeSequence}(\varepsilon_{2^X}, q) = 0$.

(9) For every set X and for every finite sequence q of elements of *Boolean* holds $\operatorname{MergeSequence}(\varepsilon_{2^X}, q) = \varepsilon_{2^X}$.

(10) For every set X and for every element x of 2^X and for every finite sequence q of elements of *Boolean* holds $\operatorname{len} \operatorname{MergeSequence}(\langle x \rangle, q) = 1$.

(11) Let X be a set, x be an element of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\operatorname{MergeSequence}(\langle x \rangle, q))(1) = x$, and

(ii) if $q(1) = \mathit{false}$, then $(\operatorname{MergeSequence}(\langle x \rangle, q))(1) = X \setminus x$.

[2]The proposition (1) has been removed.

(12) For every set X and for all elements x, y of 2^X and for every finite sequence q of elements of *Boolean* holds len MergeSequence$(\langle x, y\rangle, q) = 2$.

(13) Let X be a set, x, y be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\mathrm{MergeSequence}(\langle x, y\rangle, q))(1) = x$,

(ii) if $q(1) = \mathit{false}$, then $(\mathrm{MergeSequence}(\langle x, y\rangle, q))(1) = X \setminus x$,

(iii) if $q(2) = \mathit{true}$, then $(\mathrm{MergeSequence}(\langle x, y\rangle, q))(2) = y$, and

(iv) if $q(2) = \mathit{false}$, then $(\mathrm{MergeSequence}(\langle x, y\rangle, q))(2) = X \setminus y$.

(14) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then len MergeSequence$(\langle x, y, z\rangle, q) = 3$.

(15) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(1) = x$,

(ii) if $q(1) = \mathit{false}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(1) = X \setminus x$,

(iii) if $q(2) = \mathit{true}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(2) = y$,

(iv) if $q(2) = \mathit{false}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(2) = X \setminus y$,

(v) if $q(3) = \mathit{true}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(3) = z$, and

(vi) if $q(3) = \mathit{false}$, then $(\mathrm{MergeSequence}(\langle x, y, z\rangle, q))(3) = X \setminus z$.

(16) Let X be a set and p be a finite sequence of elements of 2^X. Then $\{\mathrm{Intersect}(\mathrm{rng}\,\mathrm{MergeSequence}(p, q)); q$ ranges over finite sequences of elements of *Boolean*: $\mathrm{len}\, q = \mathrm{len}\, p\}$ is a family of subsets of X.

Let X be a set and let Y be a finite family of subsets of X. The functor Components Y yields a family of subsets of X and is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence p of elements of 2^X such that $\mathrm{len}\, p = \mathrm{card}\, Y$ and $\mathrm{rng}\, p = Y$ and Components $Y = \{\mathrm{Intersect}(\mathrm{rng}\,\mathrm{MergeSequence}(p, q)); q$ ranges over finite sequences of elements of *Boolean*: $\mathrm{len}\, q = \mathrm{len}\, p\}$.

Let X be a set and let Y be a finite family of subsets of X. Note that Components Y is finite.

One can prove the following four propositions:

(17) For every set X and for every empty family Y of subsets of X holds Components $Y = \{X\}$.

(18) For every set X and for all finite families Y, Z of subsets of X such that $Z \subseteq Y$ holds Components Y is finer than Components Z.

(19) For every set X and for every finite family Y of subsets of X holds $\bigcup$ Components $Y = X$.

(20) Let X be a set, Y be a finite family of subsets of X, and A, B be sets. If $A \in$ Components Y and $B \in$ Components Y and $A \neq B$, then $A \cap B = \emptyset$.

Let X be a set and let Y be a finite family of subsets of X. We say that Y is in general position if and only if:

(Def. 3) $\emptyset \notin \text{Components}\, Y$.

We now state three propositions:

(21) Let X be a set and Y, Z be finite families of subsets of X. If Z is in general position and $Y \subseteq Z$, then Y is in general position.

(22) For every non empty set X holds every empty family of subsets of X is in general position.

(23) Let X be a non empty set and Y be a finite family of subsets of X. If Y is in general position, then $\text{Components}\, Y$ is a partition of X.

3. About Basis of Topological Spaces

We now state two propositions:

(24) For every non empty relational structure L holds Ω_L is infs-closed and sups-closed.

(25) For every non empty relational structure L holds Ω_L has bottom and top.

Let L be a non empty relational structure. Observe that Ω_L is infs-closed and sups-closed and has bottom and top.

The following propositions are true:

(26) For every continuous sup-semilattice L holds Ω_L is a CLbasis of L.

(27) For every up-complete non empty poset L such that L is finite holds the carrier of $L =$ the carrier of $\text{CompactSublatt}(L)$.

(28) For every lower-bounded sup-semilattice L and for every subset B of L such that B is infinite holds $\overline{\overline{B}} = \overline{\overline{\text{finsups}(B)}}$.

(29) For every T_0 non empty topological space T holds $\overline{\overline{\text{the carrier of } T}} \subseteq \overline{\overline{\text{the topology of } T}}$.

(30) Let T be a topological structure and X be a subset of T. Suppose X is open. Let B be a finite family of subsets of T. Suppose B is a basis of T. Let Y be a set. If $Y \in \text{Components}\, B$, then $X \cap Y = \emptyset$ or $Y \subseteq X$.

(31) For every T_0 topological space T such that T is infinite holds every basis of T is infinite.

(32) Let T be a non empty topological space. Suppose T is finite. Let B be a basis of T and x be an element of T. Then $\bigcap\{A; A$ ranges over elements of the topology of T: $x \in A\} \in B$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[3] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[5] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Robert Milewski. Algebraic lattices. *Formalized Mathematics*, 6(**2**):249–254, 1997.
[11] Robert Milewski. Bases of continuous lattices. *Formalized Mathematics*, 7(**2**):285–294, 1998.
[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[14] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[15] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(**2**):233–236, 1996.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[18] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[19] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[22] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.
[23] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received June 22, 1999