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Retracts and Inheritance[1]

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The notation and terminology used in this paper are introduced in the following papers: [20], [10], [8], [9], [7], [17], [1], [22], [13], [21], [18], [2], [24], [25], [23], [19], [12], [27], [15], [4], [11], [5], [3], [14], [26], [6], and [16].

1. Poset Retracts

The following three propositions are true:

(1) For all binary relations a, b holds $a \cdot b = a\,b$.

(2) Let X be a set, L be a non empty relational structure, S be a non empty relational substructure of L, f, g be functions from X into the carrier of S, and f', g' be functions from X into the carrier of L. If $f' = f$ and $g' = g$ and $f \leqslant g$, then $f' \leqslant g'$.

(3) Let X be a set, L be a non empty relational structure, S be a full non empty relational substructure of L, f, g be functions from X into the carrier of S, and f', g' be functions from X into the carrier of L. If $f' = f$ and $g' = g$ and $f' \leqslant g'$, then $f \leqslant g$.

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is directed-sups-preserving and monotone.

The following proposition is true

(4) For all functions f, g such that f is idempotent and $\mathrm{rng}\, g \subseteq \mathrm{rng}\, f$ and $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$ holds $f \cdot g = g$.

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Let S be a 1-sorted structure. Note that there exists a map from S into S which is idempotent.

One can prove the following propositions:

(5) For every up-complete non empty poset L holds every directed-sups-inheriting full non empty relational substructure of L is up-complete.

(6) Let L be an up-complete non empty poset and f be a map from L into L. Suppose f is idempotent and directed-sups-preserving. Then $\operatorname{Im} f$ is directed-sups-inheriting.

(7) Let T be an up-complete non empty poset and S be a directed-sups-inheriting full non empty relational substructure of T. Then $\operatorname{incl}(S, T)$ is directed-sups-preserving.

(8) Let S, T be non empty relational structures, f be a map from T into S, and g be a map from S into T. If $f \cdot g = \mathrm{id}_S$, then $\operatorname{rng} f =$ the carrier of S.

(9) Let T be a non empty relational structure, S be a non empty relational substructure of T, and f be a map from T into S. If $f \cdot \operatorname{incl}(S, T) = \mathrm{id}_S$, then f is an idempotent map from T into T.

Let S, T be non empty posets and let f be a function. We say that f is a retraction of T into S if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) f is a directed-sups-preserving map from T into S,

(ii) $f \restriction$the carrier of $S = \mathrm{id}_S$, and

(iii) S is a directed-sups-inheriting full relational substructure of T.

We say that f is a UPS retraction of T into S if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) f is a directed-sups-preserving map from T into S, and

(ii) there exists a directed-sups-preserving map g from S into T such that $f \cdot g = \mathrm{id}_S$.

Let S, T be non empty posets. We say that S is a retract of T if and only if:

(Def. 3) There exists a map f from T into S such that f is a retraction of T into S.

We say that S is a UPS retract of T if and only if:

(Def. 4) There exists a map f from T into S such that f is a UPS retraction of T into S.

The following propositions are true:

(10) For all non empty posets S, T and for every function f such that f is a retraction of T into S holds $f \cdot \operatorname{incl}(S, T) = \mathrm{id}_S$.

(11) Let S be a non empty poset, T be an up-complete non empty poset, and f be a function. Suppose f is a retraction of T into S. Then f is a UPS retraction of T into S.

(12) Let S, T be non empty posets and f be a function. If f is a retraction of T into S, then $\operatorname{rng} f =$ the carrier of S.

(13) Let S, T be non empty posets and f be a function. If f is a UPS retraction of T into S, then $\operatorname{rng} f =$ the carrier of S.

(14) Let S, T be non empty posets and f be a function. Suppose f is a retraction of T into S. Then f is an idempotent map from T into T.

(15) Let T, S be non empty posets and f be a map from T into T. Suppose f is a retraction of T into S. Then $\operatorname{Im} f =$ the relational structure of S.

(16) Let T be an up-complete non empty poset, S be a non empty poset, and f be a map from T into T. Suppose f is a retraction of T into S. Then f is directed-sups-preserving and projection.

(17) Let S, T be non empty reflexive transitive relational structures and f be a map from S into T. Then f is isomorphic if and only if the following conditions are satisfied:

(i) f is monotone, and

(ii) there exists a monotone map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(18) Let S, T be non empty posets. Then S and T are isomorphic if and only if there exists a monotone map f from S into T and there exists a monotone map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(19) Let S, T be up-complete non empty posets. Suppose S and T are isomorphic. Then S is a UPS retract of T and T is a UPS retract of S.

(20) Let T, S be non empty posets, f be a monotone map from T into S, and g be a monotone map from S into T. Suppose $f \cdot g = \mathrm{id}_S$. Then there exists a projection map h from T into T such that $h = g \cdot f$ and $h{\restriction}$the carrier of $\operatorname{Im} h = \mathrm{id}_{\operatorname{Im} h}$ and S and $\operatorname{Im} h$ are isomorphic.

(21) Let T be an up-complete non empty poset, S be a non empty poset, and f be a function. Suppose f is a UPS retraction of T into S. Then there exists a directed-sups-preserving projection map h from T into T such that h is a retraction of T into $\operatorname{Im} h$ and S and $\operatorname{Im} h$ are isomorphic.

(22) For every up-complete non empty poset L and for every non empty poset S such that S is a retract of L holds S is up-complete.

(23) For every complete non empty poset L and for every non empty poset S such that S is a retract of L holds S is complete.

(24) Let L be a continuous complete lattice and S be a non empty poset. If S is a retract of L, then S is continuous.

(25) Let L be an up-complete non empty poset and S be a non empty poset. If S is a UPS retract of L, then S is up-complete.

(26) Let L be a complete non empty poset and S be a non empty poset. If S is a UPS retract of L, then S is complete.

(27) Let L be a continuous complete lattice and S be a non empty poset. If S is a UPS retract of L, then S is continuous.

(28) Let L be a relational structure, S be a full relational substructure of L, and R be a relational substructure of S. Then R is full if and only if R is a full relational substructure of L.

(29) Let L be a non empty transitive relational structure and S be a directed-sups-inheriting non empty full relational substructure of L. Then every directed-sups-inheriting non empty relational substructure of S is a directed-sups-inheriting relational substructure of L.

(30) Let L be an up-complete non empty poset and S_1, S_2 be directed-sups-inheriting full non empty relational substructures of L. Suppose S_1 is a relational substructure of S_2. Then S_1 is a directed-sups-inheriting full relational substructure of S_2.

Let X, Y be non empty topological spaces. One can check that every continuous map from X into Y is continuous.

2. Products

Let R be a binary relation. We say that R is poset-yielding if and only if:

(Def. 5) For every set x such that $x \in \operatorname{rng} R$ holds x is a poset.

Let us observe that every binary relation which is poset-yielding is also relational structure yielding and reflexive-yielding.

Let X be a non empty set, let L be a non empty relational structure, let i be an element of X, and let Y be a subset of L^X. Then $\pi_i Y$ is a subset of L.

Let X be a set and let S be a poset. Note that $X \longmapsto S$ is poset-yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is poset-yielding and nonempty.

Let I be a non empty set and let J be a poset-yielding nonempty many sorted set indexed by I. Note that $\prod J$ is transitive and antisymmetric.

Let I be a non empty set, let J be a poset-yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty poset.

Next we state a number of propositions:

(31) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, f be an element of $\prod J$, and X be a subset of $\prod J$. Then $f \geqslant X$ if and only if for every element i of I holds $f(i) \geqslant \pi_i X$.

(32) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, f be an element of $\prod J$, and X be a subset of $\prod J$. Then $f \leqslant X$ if and only if for every element i of I holds $f(i) \leqslant \pi_i X$.

(33) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. Then sup X exists in $\prod J$ if and only if for every element i of I holds sup $\pi_i X$ exists in $J(i)$.

(34) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. Then inf X exists in $\prod J$ if and only if for every element i of I holds inf $\pi_i X$ exists in $J(i)$.

(35) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. If sup X exists in $\prod J$, then for every element i of I holds $(\sup X)(i) = \sup \pi_i X$.

(36) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. If inf X exists in $\prod J$, then for every element i of I holds $(\inf X)(i) = \inf \pi_i X$.

(37) Let I be a non empty set, J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I, X be a directed subset of $\prod J$, and i be an element of I. Then $\pi_i X$ is directed.

(38) Let I be a non empty set and J, K be relational structure yielding nonempty many sorted sets indexed by I. Suppose that for every element i of I holds $K(i)$ is a relational substructure of $J(i)$. Then $\prod K$ is a relational substructure of $\prod J$.

(39) Let I be a non empty set and J, K be relational structure yielding nonempty many sorted sets indexed by I. Suppose that for every element i of I holds $K(i)$ is a full relational substructure of $J(i)$. Then $\prod K$ is a full relational substructure of $\prod J$.

(40) Let L be a non empty relational structure, S be a non empty relational substructure of L, and X be a set. Then S^X is a relational substructure of L^X.

(41) Let L be a non empty relational structure, S be a full non empty relational substructure of L, and X be a set. Then S^X is a full relational substructure of L^X.

3. Inheritance

Let S, T be non empty relational structures and let X be a set. We say that S inherits sup of X from T if and only if:

(Def. 6) If sup X exists in T, then $\bigsqcup_T X \in$ the carrier of S.

We say that S inherits inf of X from T if and only if:

(Def. 7) If inf X exists in T, then $\bigsqcap_T X \in$ the carrier of S.

Next we state two propositions:

(42) Let T be a non empty transitive relational structure, S be a full non empty relational substructure of T, and X be a subset of S. Then S inherits sup of X from T if and only if if sup X exists in T, then sup X exists in S and $\sup X = \bigsqcup_T X$.

(43) Let T be a non empty transitive relational structure, S be a full non empty relational substructure of T, and X be a subset of S. Then S inherits inf of X from T if and only if if inf X exists in T, then inf X exists in S and $\inf X = \bigsqcap_T X$.

In this article we present several logical schemes. The scheme *ProductSupsInher* deals with a non empty set $\mathcal{A}$, poset-yielding nonempty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and and states that:

For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ holds $\prod \mathcal{C}$ inherits sup of X from $\prod \mathcal{B}$

provided the following conditions are satisfied:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}(i)]$,
- For every element i of $\mathcal{A}$ holds $\mathcal{C}(i)$ is a full relational substructure of $\mathcal{B}(i)$, and
- For every element i of $\mathcal{A}$ and for every subset X of $\mathcal{C}(i)$ such that $\mathcal{P}[X, \mathcal{C}(i)]$ holds $\mathcal{C}(i)$ inherits sup of X from $\mathcal{B}(i)$.

The scheme *PowerSupsInherit* deals with a non empty set $\mathcal{A}$, a non empty poset $\mathcal{B}$, a non empty full relational substructure $\mathcal{C}$ of $\mathcal{B}$, and and states that:

For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ holds $\mathcal{C}^{\mathcal{A}}$ inherits sup of X from $\mathcal{B}^{\mathcal{A}}$

provided the following requirements are met:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}]$, and
- For every subset X of $\mathcal{C}$ such that $\mathcal{P}[X, \mathcal{C}]$ holds $\mathcal{C}$ inherits sup of X from $\mathcal{B}$.

The scheme *ProductInfsInher* deals with a non empty set $\mathcal{A}$, poset-yielding nonempty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and and states that:

For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ holds $\prod \mathcal{C}$ inherits inf of X from $\prod \mathcal{B}$

provided the parameters meet the following conditions:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,

- For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}(i)]$,
- For every element i of $\mathcal{A}$ holds $\mathcal{C}(i)$ is a full relational substructure of $\mathcal{B}(i)$, and
- For every element i of $\mathcal{A}$ and for every subset X of $\mathcal{C}(i)$ such that $\mathcal{P}[X, \mathcal{C}(i)]$ holds $\mathcal{C}(i)$ inherits inf of X from $\mathcal{B}(i)$.

The scheme *PowerInfsInherit* deals with a non empty set $\mathcal{A}$, a non empty poset $\mathcal{B}$, a non empty full relational substructure $\mathcal{C}$ of $\mathcal{B}$, and and states that:

For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ holds $\mathcal{C}^{\mathcal{A}}$ inherits inf of X from $\mathcal{B}^{\mathcal{A}}$

provided the following conditions are satisfied:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}]$, and
- For every subset X of $\mathcal{C}$ such that $\mathcal{P}[X, \mathcal{C}]$ holds $\mathcal{C}$ inherits inf of X from $\mathcal{B}$.

Let I be a set, let L be a non empty relational structure, let X be a non empty subset of L^I, and let i be a set. Observe that $\pi_i X$ is non empty.

The following proposition is true

(44) Let L be a non empty poset, S be a directed-sups-inheriting non empty full relational substructure of L, and X be a non empty set. Then S^X is a directed-sups-inheriting relational substructure of L^X.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let X be a non empty subset of $\prod J$, and let i be a set. Observe that $\pi_i X$ is non empty.

One can prove the following proposition

(45) For every non empty set X and for every up-complete non empty poset L holds L^X is up-complete.

Let L be an up-complete non empty poset and let X be a non empty set. Note that L^X is up-complete.

4. Topological Retracts

Let T be a topological space. Note that the topology of T is non empty.

We now state a number of propositions:

(46) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then $\operatorname{rng} f =$ the carrier of S.

(47) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then f is idempotent.

(48) Let T be a non empty topological space and V be an open subset of T. Then $\chi_{V,\text{the carrier of } T}$ is a continuous map from T into the Sierpiński space.

(49) Let T be a non empty topological space and x, y be elements of T. Suppose that for every open subset W of T such that $y \in W$ holds $x \in W$. Then $[0 \longmapsto y, 1 \longmapsto x]$ is a continuous map from the Sierpiński space into T.

(50) Let T be a non empty topological space, x, y be elements of T, and V be an open subset of T. Suppose $x \in V$ and $y \notin V$. Then $\chi_{V,\text{the carrier of } T} \cdot [0 \longmapsto y, 1 \longmapsto x] = \mathrm{id}_{\text{the Sierpiński space}}$.

(51) Let T be a non empty 1-sorted structure, V, W be subsets of T, S be a topological augmentation of $2^1_\subseteq$, and f, g be maps from T into S. Suppose $f = \chi_{V,\text{the carrier of } T}$ and $g = \chi_{W,\text{the carrier of } T}$. Then $V \subseteq W$ if and only if $f \leqslant g$.

(52) Let L be a non empty relational structure, X be a non empty set, and R be a full non empty relational substructure of L^X. Suppose that for every set a holds a is an element of R iff there exists an element x of L such that $a = X \longmapsto x$. Then L and R are isomorphic.

(53) Let S, T be non empty topological spaces. Then S and T are homeomorphic if and only if there exists a continuous map f from S into T and there exists a continuous map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(54) Let T, S, R be non empty topological spaces, f be a map from T into S, g be a map from S into T, and h be a map from S into R. If $f \cdot g = \mathrm{id}_S$ and h is a homeomorphism, then $h \cdot f \cdot (g \cdot h^{-1}) = \mathrm{id}_R$.

(55) Let T, S, R be non empty topological spaces. Suppose S is a topological retract of T and S and R are homeomorphic. Then R is a topological retract of T.

(56) For every non empty topological space T and for every non empty subspace S of T holds $\mathrm{incl}(S, T)$ is continuous.

(57) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then $f \cdot \mathrm{incl}(S, T) = \mathrm{id}_S$.

(58) Let T be a non empty topological space and S be a non empty subspace of T. If S is a retract of T, then S is a topological retract of T.

(59) Let R, T be non empty topological spaces. Then R is a topological retract of T if and only if there exists a non empty subspace S of T such that S

is a retract of T and S and R are homeomorphic.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Quantales. *Formalized Mathematics*, 5(**1**):85–91, 1996.
[3] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[5] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[6] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[7] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Products and coproducts in categories. *Formalized Mathematics*, 2(**5**):701–709, 1991.
[11] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[12] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[13] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[14] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[15] Jarosław Gryko. Injective spaces. *Formalized Mathematics*, 7(**1**):57–62, 1998.
[16] Andrzej Kondracki. Mostowski's fundamental operations - Part I. *Formalized Mathematics*, 2(**3**):371–375, 1991.
[17] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[21] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[22] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[26] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.
[27] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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