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Basic Facts about Inaccessible and Measurable Cardinals

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Summary. Inaccessible, strongly inaccessible and measurable cardinals are defined, and it is proved that a measurable cardinal is strongly inaccessible. Filters on sets are defined, some facts related to the section about cardinals are proved. Existence of the Ulam matrix on non-limit cardinals is proved.

MML Identifier: CARD_FIL.

The notation and terminology used here are introduced in the following papers: [13], [2], [1], [5], [9], [6], [7], [3], [4], [14], [10], [12], [11], and [8].

1. Some Facts about Filters and Ideals on Sets

One can verify that there exists an ordinal number which is limit.

Let X, Y be sets. Then $X \setminus Y$ is a subset of X.

We now state the proposition

(1) For every set x and for every infinite set X holds $\overline{\overline{\{x\}}} < \overline{\overline{X}}$.

Let X be an infinite set. Observe that $\overline{\overline{X}}$ is infinite.

The scheme *ElemProp* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following condition is met:

- $\mathcal{B} \in \{y; y$ ranges over elements of $\mathcal{A} : \mathcal{P}[y]\}$.

For simplicity, we follow the rules: N is a cardinal number, M is an aleph, X is a non empty set, Y, Z, Z_1, Z_2, Y_1, Y_2 are subsets of X, and S is a subset of 2^X.

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One can prove the following proposition

(2)(i) $\{X\}$ is a non empty subset of 2^X,

(ii) $\emptyset \notin \{X\}$, and

(iii) for all Y_1, Y_2 holds if $Y_1 \in \{X\}$ and $Y_2 \in \{X\}$, then $Y_1 \cap Y_2 \in \{X\}$ and if $Y_1 \in \{X\}$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \{X\}$.

Let us consider X. A non empty subset of 2^X is said to be a filter of X if:

(Def. 1) $\emptyset \notin$ it and for all Y_1, Y_2 holds if $Y_1 \in$ it and $Y_2 \in$ it, then $Y_1 \cap Y_2 \in$ it and if $Y_1 \in$ it and $Y_1 \subseteq Y_2$, then $Y_2 \in$ it.

The following propositions are true:

(3) Let F be a set. Then F is a filter of X if and only if the following conditions are satisfied:

(i) F is a non empty subset of 2^X,

(ii) $\emptyset \notin F$, and

(iii) for all Y_1, Y_2 holds if $Y_1 \in F$ and $Y_2 \in F$, then $Y_1 \cap Y_2 \in F$ and if $Y_1 \in F$ and $Y_1 \subseteq Y_2$, then $Y_2 \in F$.

(4) $\{X\}$ is a filter of X.

In the sequel F, F_1, F_2, U_1 denote filters of X.

The following propositions are true:

(5) $X \in F$.

(6) If $Y \in F$, then $X \setminus Y \notin F$.

(7) Let I be a non empty subset of 2^X. Suppose that for every Y holds $Y \in I$ iff $Y^{\rm c} \in F$. Then $X \notin I$ and for all Y_1, Y_2 holds if $Y_1 \in I$ and $Y_2 \in I$, then $Y_1 \cup Y_2 \in I$ and if $Y_1 \in I$ and $Y_2 \subseteq Y_1$, then $Y_2 \in I$.

Let us consider X, S. We introduce $\mathrm{dual}\,S$ as a synonym of $S^{\rm c}$.

In the sequel S is a non empty subset of 2^X.

Let us consider X, S. One can verify that $S^{\rm c}$ is non empty.

One can prove the following two propositions:

(8) $\mathrm{dual}\,S = \{Y : Y^{\rm c} \in S\}$.

(9) $\mathrm{dual}\,S = \{Y^{\rm c} : Y \in S\}$.

Let us consider X. A non empty subset of 2^X is said to be an ideal of X if:

(Def. 2) $X \notin$ it and for all Y_1, Y_2 holds if $Y_1 \in$ it and $Y_2 \in$ it, then $Y_1 \cup Y_2 \in$ it and if $Y_1 \in$ it and $Y_2 \subseteq Y_1$, then $Y_2 \in$ it.

Let us consider X, F. Then $\mathrm{dual}\,F$ is an ideal of X.

In the sequel I is an ideal of X.

Next we state two propositions:

(10) For every Y holds $Y \notin F$ or $Y \notin \mathrm{dual}\,F$ and for every Y holds $Y \notin I$ or $Y \notin \mathrm{dual}\,I$.

(11) $\emptyset \in I$.

Let us consider X, N, S. We say that S is multiplicative with N if and only if:

(Def. 3) For every non empty set S_1 such that $S_1 \subseteq S$ and $\overline{\overline{S_1}} < N$ holds $\bigcap S_1 \in S$.

Let us consider X, N, S. We say that S is additive with N if and only if:

(Def. 4) For every non empty set S_1 such that $S_1 \subseteq S$ and $\overline{\overline{S_1}} < N$ holds $\bigcup S_1 \in S$.

Let us consider X, N, F. We introduce F is complete with N as a synonym of F is multiplicative with N.

Let us consider X, N, I. We introduce I is complete with N as a synonym of I is additive with N.

One can prove the following proposition

(12) If S is multiplicative with N, then dual S is additive with N.

Let us consider X, F. We say that F is uniform if and only if:

(Def. 5) For every Y such that $Y \in F$ holds $\overline{\overline{Y}} = \overline{\overline{X}}$.

We say that F is principal if and only if:

(Def. 6) There exists Y such that $Y \in F$ and for every Z such that $Z \in F$ holds $Y \subseteq Z$.

We say that F is an ultrafilter if and only if:

(Def. 7) For every Y holds $Y \in F$ or $X \setminus Y \in F$.

Let us consider X, F, Z. The functor Extend_Filter(F, Z) yields a non empty subset of 2^X and is defined as follows:

(Def. 8) Extend_Filter$(F, Z) = \{Y : \bigvee_{Y_2} (Y_2 \in \{Y_1 \cap Z : Y_1 \in F\} \ \wedge \ Y_2 \subseteq Y)\}$.

We now state two propositions:

(13) For every Z_1 holds $Z_1 \in$ Extend_Filter(F, Z) iff there exists Z_2 such that $Z_2 \in F$ and $Z_2 \cap Z \subseteq Z_1$.

(14) If for every Y_1 such that $Y_1 \in F$ holds $Y_1 \cap Z \neq \emptyset$, then $Z \in$ Extend_Filter(F, Z) and Extend_Filter(F, Z) is a filter of X and $F \subseteq$ Extend_Filter(F, Z).

In the sequel S denotes a subset of 2^X.

Let us consider X. The functor Filters X yielding a non empty subset of 2^{2^X} is defined by:

(Def. 9) Filters $X = \{S : S$ is a filter of $X\}$.

We now state the proposition

(15) For every set S holds $S \in$ Filters X iff S is a filter of X.

In the sequel F_3 is a non empty subset of Filters X.

One can prove the following propositions:

(16) If for all F_1, F_2 such that $F_1 \in F_3$ and $F_2 \in F_3$ holds $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$, then $\bigcup F_3$ is a filter of X.

(17) For every F there exists U_1 such that $F \subseteq U_1$ and U_1 is an ultrafilter.

In the sequel X denotes an infinite set, Y denotes a subset of X, and F, U_1 denote filters of X.

Let us consider X. The functor Frechet_Filter X yielding a filter of X is defined by:

(Def. 10) Frechet_Filter $X = \{Y : \overline{\overline{X \setminus Y}} < \overline{\overline{X}}\}$.

Let us consider X. The functor Frechet_Ideal X yields an ideal of X and is defined as follows:

(Def. 11) Frechet_Ideal $X =$ dual Frechet_Filter X.

One can prove the following propositions:

(18) $Y \in$ Frechet_Filter X iff $\overline{\overline{X \setminus Y}} < \overline{\overline{X}}$.

(19) $Y \in$ Frechet_Ideal X iff $\overline{\overline{Y}} < \overline{\overline{X}}$.

(20) If Frechet_Filter $X \subseteq F$, then F is uniform.

(21) If U_1 is uniform and an ultrafilter, then Frechet_Filter $X \subseteq U_1$.

Let us consider X. One can check that there exists a filter of X which is non principal and an ultrafilter.

Let us consider X. One can check that every filter of X which is uniform and an ultrafilter is also non principal.

Next we state two propositions:

(22) For every an ultrafilter filter F of X and for every Y holds $Y \in F$ iff $Y \notin$ dual F.

(23) If F is non principal and an ultrafilter and F is complete with $\overline{\overline{X}}$, then F is uniform.

2. Inaccessible and Measurable Cardinals, Ulam Matrix

We now state the proposition

(24) $N^+ \leqslant \overline{\mathbf{2}}^N$.

We say that Generalized Continuum Hypothesis holds if and only if:

(Def. 12) For every N holds $N^+ = \overline{\mathbf{2}}^N$.

Let I_1 be an aleph. We say that I_1 is inaccessible if and only if:

(Def. 13) I_1 is regular and limit.

We introduce I_1 is inaccessible cardinal as a synonym of I_1 is inaccessible.

Let us note that every aleph which is inaccessible is also regular and limit.

We now state the proposition

(25) $\aleph_0$ is inaccessible.

Let I_1 be an aleph. We say that I_1 is strong limit if and only if:

(Def. 14) For every N such that $N < I_1$ holds $\overline{\mathbf{2}}^N < I_1$.

We introduce I_1 is strong limit cardinal as a synonym of I_1 is strong limit.

Next we state two propositions:

(26) $\aleph_0$ is strong limit.

(27) If M is strong limit, then M is limit.

One can check that every aleph which is strong limit is also limit.

Next we state the proposition

(28) If Generalized Continuum Hypothesis holds, then if M is limit, then M is strong limit.

Let I_1 be an aleph. We say that I_1 is strongly inaccessible if and only if:

(Def. 15) I_1 is regular and strong limit.

We introduce I_1 is strongly inaccessible cardinal as a synonym of I_1 is strongly inaccessible.

Let us observe that every aleph which is strongly inaccessible is also regular and strong limit.

The following propositions are true:

(29) $\aleph_0$ is strongly inaccessible.

(30) If M is strongly inaccessible, then M is inaccessible.

Let us note that every aleph which is strongly inaccessible is also inaccessible.

Next we state the proposition

(31) If Generalized Continuum Hypothesis holds, then if M is inaccessible, then M is strongly inaccessible.

Let us consider M. We say that M is measurable if and only if:

(Def. 16) There exists a filter U_1 of M such that U_1 is complete with M and U_1 is non principal and an ultrafilter.

We introduce M is measurable cardinal as a synonym of M is measurable.

We now state two propositions:

(32) For every limit ordinal number A and for every set X such that $X \subseteq A$ holds if $\sup X = A$, then $\bigcup X = A$.

(33) If M is measurable, then M is regular.

Let us consider M. Note that M^+ is non limit.

Let us note that there exists a cardinal number which is non limit and infinite.

Let us observe that every aleph which is non limit is also regular.

Let M be a non limit cardinal number. The functor predecessor M yields a cardinal number and is defined as follows:

(Def. 17) $M = (\mathrm{predecessor}\, M)^+$.

Let M be a non limit aleph. One can check that predecessor M is infinite.

Let X be a set and let N, N_1 be cardinal numbers. An Inf Matrix of N, N_1, X is a function from $[\!:N, N_1:\!]$ into X.

For simplicity, we follow the rules: X denotes a set, M denotes a non limit aleph, F denotes a filter of M, N_1, N_2 denote elements of predecessor M, K_1, K_2 denote elements of M, and T denotes an Inf Matrix of predecessor M, M, 2^M.

Let us consider M, T. We say that T is Ulam Matrix of M if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) For all N_1, K_1, K_2 such that $K_1 \neq K_2$ holds $T(N_1, K_1) \cap T(N_1, K_2)$ is empty,

(ii) for all K_1, N_1, N_2 such that $N_1 \neq N_2$ holds $T(N_1, K_1) \cap T(N_2, K_1)$ is empty,

(iii) for every N_1 holds $\overline{\overline{M \setminus \bigcup\{T(N_1, K_1) : K_1 \in M\}}} \leqslant$ predecessor M, and

(iv) for every K_1 holds $\overline{\overline{M \setminus \bigcup\{T(N_1, K_1) : N_1 \in \text{predecessor}\, M\}}} \leqslant$ predecessor M.

The following four propositions are true:

(34) There exists T such that T is Ulam Matrix of M.

(35) Let given M and I be an ideal of M. Suppose I is complete with M and Frechet_Ideal $M \subseteq I$. Then there exists a subset S of 2^M such that $\overline{\overline{S}} = M$ and for every set X_1 such that $X_1 \in S$ holds $X_1 \notin I$ and for all sets X_1, X_2 such that $X_1 \in S$ and $X_2 \in S$ and $X_1 \neq X_2$ holds $X_1 \cap X_2 = \emptyset$.

(36) For every X and for every cardinal number N such that $N \leqslant \overline{\overline{X}}$ there exists a set Y such that $Y \subseteq X$ and $\overline{\overline{Y}} = N$.

(37) For every M it is not true that there exists F such that F is uniform and an ultrafilter and F is complete with M.

In the sequel M is an aleph.

The following four propositions are true:

(38) If M is measurable, then M is limit.

(39) If M is measurable, then M is inaccessible.

(40) If M is measurable, then M is strong limit.

(41) If M is measurable, then M is strongly inaccessible.

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