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The Hahn Banach Theorem in the Vector Space over the Field of Complex Numbers

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Summary. This article contains the Hahn Banach theorem in the vector space over the field of complex numbers.

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The articles [8], [7], [1], [5], [2], [6], [9], [3], [14], [10], [12], [13], [4], and [11] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For every element z of $\mathbb{C}$ holds $||z|| = |z|$.

(2) For all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ holds $(x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 \cdot x_2 - y_1 \cdot y_2) + (x_1 \cdot y_2 + x_2 \cdot y_1)i$.

(3) For every real number r holds $(r + 0i) \cdot i = 0 + ri$.

(4) For every real number r holds $|r + 0i| = |r|$.

(5) For every element z of $\mathbb{C}$ such that $|z| \neq 0$ holds $|z| + 0i = \frac{z^*}{|z|+0i} \cdot z$.

2. Some Facts on the Field of Complex Numbers

Let x, y be real numbers. The functor $x + yi_{\mathbb{C}_{\mathrm{F}}}$ yielding an element of $\mathbb{C}_{\mathrm{F}}$ is defined by:

(Def. 1) $x + yi_{\mathbb{C}_{\mathrm{F}}} = x + yi$.

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The element $i_{\mathbb{C}_{\mathrm{F}}}$ of $\mathbb{C}_{\mathrm{F}}$ is defined by:

(Def. 2) $i_{\mathbb{C}_{\mathrm{F}}} = i$.

One can prove the following propositions:

(6) $i_{\mathbb{C}_{\mathrm{F}}} = 0 + 1i$ and $i_{\mathbb{C}_{\mathrm{F}}} = 0 + 1i_{\mathbb{C}_{\mathrm{F}}}$.

(7) $|i_{\mathbb{C}_{\mathrm{F}}}| = 1$.

(8) $i_{\mathbb{C}_{\mathrm{F}}} \cdot i_{\mathbb{C}_{\mathrm{F}}} = -\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(9) $(-\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}) \cdot -\mathbf{1}_{\mathbb{C}_{\mathrm{F}}} = \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(10) For all real numbers x_1, y_1, x_2, y_2 holds $(x_1 + y_1 i_{\mathbb{C}_{\mathrm{F}}}) + (x_2 + y_2 i_{\mathbb{C}_{\mathrm{F}}}) = (x_1 + x_2) + (y_1 + y_2) i_{\mathbb{C}_{\mathrm{F}}}$.

(11) For all real numbers x_1, y_1, x_2, y_2 holds $(x_1 + y_1 i_{\mathbb{C}_{\mathrm{F}}}) \cdot (x_2 + y_2 i_{\mathbb{C}_{\mathrm{F}}}) = (x_1 \cdot x_2 - y_1 \cdot y_2) + (x_1 \cdot y_2 + x_2 \cdot y_1) i_{\mathbb{C}_{\mathrm{F}}}$.

(12) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $||z|| = |z|$.

(13) For every real number r holds $|r + 0 i_{\mathbb{C}_{\mathrm{F}}}| = |r|$.

(14) For every real number r holds $(r + 0 i_{\mathbb{C}_{\mathrm{F}}}) \cdot i_{\mathbb{C}_{\mathrm{F}}} = 0 + r i_{\mathbb{C}_{\mathrm{F}}}$.

Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $\Re(z)$ yields a real number and is defined as follows:

(Def. 3) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $\Re(z) = \Re(z')$.

Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $\Im(z)$ yields a real number and is defined as follows:

(Def. 4) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $\Im(z) = \Im(z')$.

The following propositions are true:

(15) For all real numbers x, y holds $\Re(x + y i_{\mathbb{C}_{\mathrm{F}}}) = x$ and $\Im(x + y i_{\mathbb{C}_{\mathrm{F}}}) = y$.

(16) For all elements x, y of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(x + y) = \Re(x) + \Re(y)$ and $\Im(x + y) = \Im(x) + \Im(y)$.

(17) For all elements x, y of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(x \cdot y) = \Re(x) \cdot \Re(y) - \Im(x) \cdot \Im(y)$ and $\Im(x \cdot y) = \Re(x) \cdot \Im(y) + \Re(y) \cdot \Im(x)$.

(18) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(z) \leqslant |z|$.

(19) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Im(z) \leqslant |z|$.

3. Functionals of Vector Space

Let K be a 1-sorted structure and let V be a vector space structure over K.

(Def. 5) A function from the carrier of V into the carrier of K is said to be a functional in V.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be functionals in V. The functor $f + g$ yielding a functional in V is defined by:

(Def. 6) For every element x of the carrier of V holds $(f + g)(x) = f(x) + g(x)$.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f be a functional in V. The functor $-f$ yielding a functional in V is defined by:

(Def. 7) For every element x of the carrier of V holds $(-f)(x) = -f(x)$.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be functionals in V. The functor $f - g$ yielding a functional in V is defined by:

(Def. 8) $f - g = f + -g$.

Let K be a non empty groupoid, let V be a non empty vector space structure over K, let v be an element of the carrier of K, and let f be a functional in V. The functor $v \cdot f$ yields a functional in V and is defined by:

(Def. 9) For every element x of the carrier of V holds $(v \cdot f)(x) = v \cdot f(x)$.

Let K be a non empty zero structure and let V be a vector space structure over K. The functor 0Functional V yields a functional in V and is defined as follows:

(Def. 10) 0Functional $V = \Omega_V \longmapsto 0_K$.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let F be a functional in V. We say that F is additive if and only if:

(Def. 11) For all vectors x, y of V holds $F(x + y) = F(x) + F(y)$.

Let K be a non empty groupoid, let V be a non empty vector space structure over K, and let F be a functional in V. We say that F is homogeneous if and only if:

(Def. 12) For every vector x of V and for every scalar r of V holds $F(r \cdot x) = r \cdot F(x)$.

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let F be a functional in V. We say that F is 0-preserving if and only if:

(Def. 13) $F(0_V) = 0_K$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure and let V be a vector space over K. Note that every functional in V which is homogeneous is also 0-preserving.

Let K be a right zeroed non empty loop structure and let V be a non empty vector space structure over K. Note that 0Functional V is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V be a non empty vector space structure over K. Observe that 0Functional V is homogeneous.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that 0Functional V is 0-preserving.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V be a non empty vector space structure over K. Observe that there exists a functional in V which is additive, homogeneous, and 0-preserving.

The following propositions are true:

(20) Let K be an Abelian non empty loop structure, V be a non empty vector space structure over K, and f, g be functionals in V. Then $f+g=g+f$.

(21) Let K be an add-associative non empty loop structure, V be a non empty vector space structure over K, and f, g, h be functionals in V. Then $(f+g)+h=f+(g+h)$.

(22) Let K be a non empty zero structure, V be a non empty vector space structure over K, and x be an element of the carrier of V. Then $(0\mathrm{Functional}\,V)(x)=0_K$.

(23) Let K be a right zeroed non empty loop structure, V be a non empty vector space structure over K, and f be a functional in V. Then $f+0\mathrm{Functional}\,V=f$.

(24) Let K be an add-associative right zeroed right complementable non empty loop structure, V be a non empty vector space structure over K, and f be a functional in V. Then $f-f=0\mathrm{Functional}\,V$.

(25) Let K be a right distributive non empty double loop structure, V be a non empty vector space structure over K, r be an element of the carrier of K, and f, g be functionals in V. Then $r\cdot(f+g)=r\cdot f+r\cdot g$.

(26) Let K be a left distributive non empty double loop structure, V be a non empty vector space structure over K, r, s be elements of the carrier of K, and f be a functional in V. Then $(r+s)\cdot f=r\cdot f+s\cdot f$.

(27) Let K be an associative non empty groupoid, V be a non empty vector space structure over K, r, s be elements of the carrier of K, and f be a functional in V. Then $(r\cdot s)\cdot f=r\cdot(s\cdot f)$.

(28) Let K be a left unital non empty double loop structure, V be a non empty vector space structure over K, and f be a functional in V. Then $\mathbf{1}_K\cdot f=f$.

Let K be an Abelian add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, and let f, g be additive functionals in V. Observe that $f+g$ is additive.

Let K be an Abelian add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, and let f be an additive functional in V. One can verify that $-f$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, let v be an element of the carrier of K, and let f be an additive functional in V. Observe that $v \cdot f$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, and let f, g be homogeneous functionals in V. Observe that $f + g$ is homogeneous.

Let K be an Abelian add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, and let f be a homogeneous functional in V. One can check that $-f$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive associative commutative non empty double loop structure, let V be a non empty vector space structure over K, let v be an element of the carrier of K, and let f be a homogeneous functional in V. Observe that $v \cdot f$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V be a non empty vector space structure over K. A linear functional in V is an additive homogeneous functional in V.

4. The Vector Space of Linear Functionals

Let K be an Abelian add-associative right zeroed right complementable right distributive associative commutative non empty double loop structure and let V be a non empty vector space structure over K. The functor V^* yielding a non empty strict vector space structure over K is defined by the conditions (Def. 14).

(Def. 14)(i) For every set x holds $x \in$ the carrier of V^* iff x is a linear functional in V,

(ii) for all linear functionals f, g in V holds (the addition of V^*)$(f, g) = f + g$,

(iii) for every linear functional f in V holds (the reverse-map of V^*)$(f) = -f$,

(iv) the zero of $V^* = 0\mathrm{Functional}\,V$, and

(v) for every linear functional f in V and for every element x of the carrier of K holds (the left multiplication of V^*)$(x, f) = x \cdot f$.

Let K be an Abelian add-associative right zeroed right complementable right distributive associative commutative non empty double loop structure and let V be a non empty vector space structure over K. One can check that V^* is Abelian.

Let K be an Abelian add-associative right zeroed right complementable right distributive associative commutative non empty double loop structure and let

V be a non empty vector space structure over K. One can verify the following observations:

* V^* is add-associative,
* V^* is right zeroed, and
* V^* is right complemented.

Let K be an Abelian add-associative right zeroed right complementable left unital distributive associative commutative non empty double loop structure and let V be a non empty vector space structure over K. One can check that V^* is vector space-like.

5. Semi Norm of Vector Space

Let K be a 1-sorted structure and let V be a vector space structure over K.

(Def. 15) A function from the carrier of V into $\mathbb{R}$ is said to be a RFunctional of V.

Let K be a 1-sorted structure, let V be a non empty vector space structure over K, and let F be a RFunctional of V. We say that F is subadditive if and only if:

(Def. 16) For all vectors x, y of V holds $F(x+y) \leqslant F(x)+F(y)$.

Let K be a 1-sorted structure, let V be a non empty vector space structure over K, and let F be a RFunctional of V. We say that F is additive if and only if:

(Def. 17) For all vectors x, y of V holds $F(x+y) = F(x)+F(y)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let F be a RFunctional of V. We say that F is Real-homogeneous if and only if:

(Def. 18) For every vector v of V and for every real number r holds $F((r+0i_{\mathbb{C}_{\rm F}})\cdot v) = r \cdot F(v)$.

One can prove the following proposition

(29) Let V be a vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$ and F be a RFunctional of V. Suppose F is Real-homogeneous. Let v be a vector of V and r be a real number. Then $F((0+ri_{\mathbb{C}_{\rm F}})\cdot v) = r \cdot F(i_{\mathbb{C}_{\rm F}} \cdot v)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let F be a RFunctional of V. We say that F is homogeneous if and only if:

(Def. 19) For every vector v of V and for every scalar r of V holds $F(r \cdot v) = |r| \cdot F(v)$.

Let K be a 1-sorted structure, let V be a vector space structure over K, and let F be a RFunctional of V. We say that F is 0-preserving if and only if:

(Def. 20) $F(0_V) = 0$.

Let K be a 1-sorted structure and let V be a non empty vector space structure over K. One can verify that every RFunctional of V which is additive is also subadditive.

Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$. Note that every RFunctional of V which is Real-homogeneous is also 0-preserving.

Let K be a 1-sorted structure and let V be a vector space structure over K. The functor $0\mathrm{RFunctional}\, V$ yielding a RFunctional of V is defined as follows:

(Def. 21) $0\mathrm{RFunctional}\, V = \Omega_V \longmapsto 0$.

Let K be a 1-sorted structure and let V be a non empty vector space structure over K. Note that $0\mathrm{RFunctional}\, V$ is additive and $0\mathrm{RFunctional}\, V$ is 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Note that $0\mathrm{RFunctional}\, V$ is Real-homogeneous and $0\mathrm{RFunctional}\, V$ is homogeneous.

Let K be a 1-sorted structure and let V be a non empty vector space structure over K. Note that there exists a RFunctional of V which is additive and 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a RFunctional of V which is additive, Real-homogeneous, and homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. A Semi-Norm of V is a subadditive homogeneous RFunctional of V.

6. The Hahn Banach Theorem

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. The functor $\mathrm{RealVS}\, V$ yielding a strict RLS structure is defined by the conditions (Def. 22).

(Def. 22)(i) The loop structure of $\mathrm{RealVS}\, V =$ the loop structure of V, and

(ii) for every real number r and for every vector v of V holds (the external multiplication of $\mathrm{RealVS}\, V)(r, v) = (r + 0 i_{\mathbb{C}_{\mathrm{F}}}) \cdot v$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that $\mathrm{RealVS}\, V$ is non empty.

Let V be an Abelian non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that $\mathrm{RealVS}\, V$ is Abelian.

Let V be an add-associative non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that $\mathrm{RealVS}\, V$ is add-associative.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Note that $\mathrm{RealVS}\, V$ is right zeroed.

Let V be a right complementable non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that $\mathrm{RealVS}\, V$ is right complementable.

Let V be a vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that RealVS V is real linear space-like.

One can prove the following three propositions:

(30) For every non empty vector space V over $\mathbb{C}_{\rm F}$ and for every subspace M of V holds RealVS M is a subspace of RealVS V.

(31) For every non empty vector space structure V over $\mathbb{C}_{\rm F}$ holds every RFunctional of V is a functional in RealVS V.

(32) For every non empty vector space V over $\mathbb{C}_{\rm F}$ holds every Semi-Norm of V is a Banach functional in RealVS V.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a functional in V. The functor projRe l yielding a functional in RealVS V is defined by:

(Def. 23) For every element i of the carrier of V holds $(\text{projRe}\, l)(i) = \Re(l(i))$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a functional in V. The functor projIm l yields a functional in RealVS V and is defined as follows:

(Def. 24) For every element i of the carrier of V holds $(\text{projIm}\, l)(i) = \Im(l(i))$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a functional in RealVS V. The functor $l_{\mathbb{R}\to\mathbb{C}}$ yielding a RFunctional of V is defined by:

(Def. 25) $l_{\mathbb{R}\to\mathbb{C}} = l$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a RFunctional of V. The functor $l_{\mathbb{C}\to\mathbb{R}}$ yields a functional in RealVS V and is defined by:

(Def. 26) $l_{\mathbb{C}\to\mathbb{R}} = l$.

Let V be a non empty vector space over $\mathbb{C}_{\rm F}$ and let l be an additive functional in RealVS V. One can check that $l_{\mathbb{R}\to\mathbb{C}}$ is additive.

Let V be a non empty vector space over $\mathbb{C}_{\rm F}$ and let l be an additive RFunctional of V. Observe that $l_{\mathbb{C}\to\mathbb{R}}$ is additive.

Let V be a non empty vector space over $\mathbb{C}_{\rm F}$ and let l be a homogeneous functional in RealVS V. Observe that $l_{\mathbb{R}\to\mathbb{C}}$ is Real-homogeneous.

Let V be a non empty vector space over $\mathbb{C}_{\rm F}$ and let l be a Real-homogeneous RFunctional of V. One can verify that $l_{\mathbb{C}\to\mathbb{R}}$ is homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a RFunctional of V. The functor i-shift l yields a RFunctional of V and is defined by:

(Def. 27) For every element v of the carrier of V holds $(\text{i-shift}\, l)(v) = l(i_{\mathbb{C}_{\rm F}} \cdot v)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let l be a functional in RealVS V. The functor prodReIm l yielding a functional in V is defined as follows:

(Def. 28) For every element v of the carrier of V holds $(\text{prodReIm}\, l)(v) = (l_{\mathbb{R}\to\mathbb{C}})(v) + (-(\text{i-shift}\, l_{\mathbb{R}\to\mathbb{C}})(v))i_{\mathbb{C}_{\rm F}}$.

The following four propositions are true:

(33) Let V be a non empty vector space over $\mathbb{C}_{\mathrm{F}}$ and l be a linear functional in V. Then projRe l is a linear functional in RealVS V.

(34) Let V be a non empty vector space over $\mathbb{C}_{\mathrm{F}}$ and l be a linear functional in V. Then projIm l is a linear functional in RealVS V.

(35) Let V be a non empty vector space over $\mathbb{C}_{\mathrm{F}}$ and l be a linear functional in RealVS V. Then prodReIm l is a linear functional in V.

(36) Let V be a non empty vector space over $\mathbb{C}_{\mathrm{F}}$, p be a Semi-Norm of V, M be a subspace of V, and l be a linear functional in M. Suppose that for every vector e of M and for every vector v of V such that $v = e$ holds $|l(e)| \leqslant p(v)$. Then there exists a linear functional L in V such that $L\restriction$the carrier of $M = l$ and for every vector e of V holds $|L(e)| \leqslant p(e)$.

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