FORMALIZED MATHEMATICS
Volume 9, Number 2, 2001
University of Białystok

Definition of Integrability for Partial Functions from $\mathbb{R}$ to $\mathbb{R}$ and Integrability for Continuous Functions

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Summary. In this article, we defined the Riemann definite integral of partial function from $\mathbb{R}$ to $\mathbb{R}$. Then we have proved the integrability for the continuous function and differentiable function. Moreover, we have proved an elementary theorem of calculus.

MML Identifier: `INTEGRA5`.

The articles [12], [13], [1], [2], [6], [3], [5], [14], [7], [16], [9], [10], [4], [11], [8], and [15] provide the notation and terminology for this paper.

1. Some Useful Properties of Finite Sequence

For simplicity, we adopt the following convention: i denotes a natural number, a, b, r_1, r_2 denote real numbers, A denotes a closed-interval subset of $\mathbb{R}$, C denotes a non empty set, and X denotes a set.

One can prove the following propositions:

(1) Let F, F_1, F_2 be finite sequences of elements of $\mathbb{R}$ and given r_1, r_2. If $F_1 = \langle r_1 \rangle ^\frown F$ or $F_1 = F ^\frown \langle r_1 \rangle$ and if $F_2 = \langle r_2 \rangle ^\frown F$ or $F_2 = F ^\frown \langle r_2 \rangle$, then $\sum (F_1 - F_2) = r_1 - r_2$.

(2) Let F_1, F_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} F_1 = \operatorname{len} F_2$, then $\operatorname{len}(F_1 + F_2) = \operatorname{len} F_1$ and $\operatorname{len}(F_1 - F_2) = \operatorname{len} F_1$ and $\sum (F_1 + F_2) = \sum F_1 + \sum F_2$ and $\sum (F_1 - F_2) = \sum F_1 - \sum F_2$.

(3) Let F_1, F_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} F_1 = \operatorname{len} F_2$ and for every i such that $i \in \operatorname{dom} F_1$ holds $F_1(i) \leqslant F_2(i)$, then $\sum F_1 \leqslant \sum F_2$.

ISSN 1426–2630

2. Integrability for Partial Function of ℝ, ℝ

Let C be a non empty subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $f \restriction C$ yielding a partial function from C to $\mathbb{R}$ is defined as follows:

(Def. 1) $f \restriction C = f \restriction C$.

Next we state two propositions:

(4) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $(f \restriction C)\,(g \restriction C) = (f\,g) \restriction C$.

(5) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $(f + g) \restriction C = f \restriction C + g \restriction C$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is integrable on A if and only if:

(Def. 2) $f \restriction A$ is integrable on A.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\int\limits_A f(x)dx$ yields a real number and is defined by:

(Def. 3) $\int\limits_A f(x)dx = \mathrm{integral}\, f \restriction A$.

The following propositions are true:

(6) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $A \subseteq \mathrm{dom}\, f$ holds $f \restriction A$ is total.

(7) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is upper bounded on A holds $f \restriction A$ is upper bounded on A.

(8) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is lower bounded on A holds $f \restriction A$ is lower bounded on A.

(9) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is bounded on A holds $f \restriction A$ is bounded on A.

3. Integrability for Continuous Function

The following propositions are true:

(10) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is continuous on A holds f is bounded on A.

(11) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is continuous on A holds f is integrable on A.

(12) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and D be an element of $\mathrm{divs}\, A$. Suppose $A \subseteq X$ and f is differentiable on X and $f'_{\restriction X}$ is bounded on A. Then $\mathrm{lower\_sum}(f'_{\restriction X} \restriction A, D) \leqslant f(\sup A) - f(\inf A)$ and $f(\sup A) - f(\inf A) \leqslant \mathrm{upper\_sum}(f'_{\restriction X} \restriction A, D)$.

(13) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A \subseteq X$ and f is differentiable on X and $f'_{\restriction X}$ is integrable on A and $f'_{\restriction X}$ is bounded on A. Then $\int\limits_{A} f'_{\restriction X}(x)dx = f(\sup A) - f(\inf A)$.

(14) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is non-decreasing on A and $A \subseteq \operatorname{dom} f$ holds $\operatorname{rng}(f \restriction A)$ is bounded.

(15) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If f is non-decreasing on A and $A \subseteq \operatorname{dom} f$, then $\inf \operatorname{rng}(f \restriction A) = f(\inf A)$ and $\sup \operatorname{rng}(f \restriction A) = f(\sup A)$.

(16) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is monotone on A and $A \subseteq \operatorname{dom} f$ holds f is integrable on A.

(17) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A, B be closed-interval subsets of $\mathbb{R}$. If f is continuous on A and $B \subseteq A$, then f is integrable on B.

(18) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, A, B, C be closed-interval subsets of $\mathbb{R}$, and given X. Suppose $A \subseteq X$ and f is differentiable on X and $f'_{\restriction X}$ is continuous on A and $\inf A = \inf B$ and $\sup B = \inf C$ and $\sup C = \sup A$. Then $B \subseteq A$ and $C \subseteq A$ and $\int\limits_{A} f'_{\restriction X}(x)dx = \int\limits_{B} f'_{\restriction X}(x)dx + \int\limits_{C} f'_{\restriction X}(x)dx$.

Let a, b be elements of $\mathbb{R}$. Let us assume that $a \leqslant b$. The functor $['a, b']$ yields a closed-interval subset of $\mathbb{R}$ and is defined as follows:

(Def. 4) $['a, b'] = [a, b]$.

Let a, b be elements of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\int\limits_{a}^{b} f(x)dx$ yields a real number and is defined by:

(Def. 5) $$\int\limits_{a}^{b} f(x)dx = \begin{cases} \int\limits_{['a,b']} f(x)dx, & \text{if } a \leqslant b, \\ -\int\limits_{['b,a']} f(x)dx, & \text{otherwise.} \end{cases}$$

We now state three propositions:

(19) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, A be a closed-interval subset of $\mathbb{R}$, and given a, b. If $A = [a, b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(20) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, A be a closed-interval subset of

$\mathbb{R}$, and given a, b. If $A = [b, a]$, then $-\int\limits_A f(x)dx = \int\limits_a^b f(x)dx$.

(21) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and X be an open subset of $\mathbb{R}$. Suppose that f is differentiable on X and g is differentiable on X and $A \subseteq X$ and $f'_{\restriction X}$ is integrable on A and $f'_{\restriction X}$ is bounded on A and $g'_{\restriction X}$ is integrable on A and $g'_{\restriction X}$ is bounded on A. Then $\int\limits_A f'_{\restriction X}\, g(x)dx = f(\sup A) \cdot g(\sup A) - f(\inf A) \cdot g(\inf A) - \int\limits_A f\, g'_{\restriction X}(x)dx$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[5] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[6] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[9] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.

[10] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[12] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[13] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[14] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received March 23, 2000