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Representation Theorem for Finite Distributive Lattices

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Summary. In the article the representation theorem for finite distributive lattice as rings of sets is presented. Auxiliary concepts are introduced. Namely, the concept of the height of an element, the maximal element in a chain, immediate predecessor of an element and ring of sets. Besides the schemes of induction in finite lattice is proved.

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The notation and terminology used here are introduced in the following papers: [7], [1], [8], [6], [9], [3], [4], [2], and [5].

1. Induction in a Finite Lattice

Let L be a 1-sorted structure and let A, B be subsets of L. Let us observe that $A \subseteq B$ if and only if:

(Def. 1) For every element x of L such that $x \in A$ holds $x \in B$.

Let L be a lattice. Note that there exists a chain of L which is non empty.

Let L be a lattice and let x, y be elements of L. Let us assume that $x \leqslant y$. A non empty chain of L is called a chain of x, y if:

(Def. 2) $x \in$ it and $y \in$ it and for every element z of L such that $z \in$ it holds $x \leqslant z$ and $z \leqslant y$.

The following proposition is true

(1) For every lattice L and for all elements x, y of L such that $x \leqslant y$ holds $\{x, y\}$ is a chain of x, y.

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Let L be a finite lattice and let x be an element of L. The functor height x yields a natural number and is defined as follows:

(Def. 3) There exists a chain A of $\bot_L$, x such that height x = card A and for every chain A of $\bot_L$, x holds card $A \leqslant$ height x.

Next we state several propositions:

(2) For every finite lattice L and for all elements a, b of L such that $a < b$ holds height $a <$ height b.

(3) Let L be a finite lattice, C be a chain of L, and x, y be elements of L. If $x \in C$ and $y \in C$, then $x < y$ iff height $x <$ height y.

(4) Let L be a finite lattice, C be a chain of L, and x, y be elements of L. If $x \in C$ and $y \in C$, then $x = y$ iff height $x =$ height y.

(5) Let L be a finite lattice, C be a chain of L, and x, y be elements of L. If $x \in C$ and $y \in C$, then $x \leqslant y$ iff height $x \leqslant$ height y.

(6) For every finite lattice L and for every element x of L holds height $x = 1$ iff $x = \bot_L$.

(7) For every non empty finite lattice L and for every element x of L holds height $x \geqslant 1$.

The scheme *LattInd* deals with a finite lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element x of $\mathcal{A}$ holds $\mathcal{P}[x]$

provided the following requirement is met:

- For every element x of $\mathcal{A}$ such that for every element b of $\mathcal{A}$ such that $b < x$ holds $\mathcal{P}[b]$ holds $\mathcal{P}[x]$.

2. Join Irreducible Elements in a Finite Distributive Lattice

Let us mention that there exists a lattice which is distributive and finite.

Let L be a lattice and let x, y be elements of L. The predicate $x <_1 y$ is defined as follows:

(Def. 4) $x < y$ and it is not true that there exists an element z of L such that $x < z$ and $z < y$.

One can prove the following proposition

(8) Let L be a finite lattice and X be a non empty subset of L. Then there exists an element x of L such that $x \in X$ and for every element y of L such that $y \in X$ holds $x \not< y$.

Let L be a finite lattice and let A be a non empty chain of L. The functor $\max A$ yielding an element of L is defined by:

(Def. 5) For every element x of L such that $x \in A$ holds $x \leqslant \max A$ and $\max A \in A$.

The following proposition is true

(9) For every finite lattice L and for every element y of L such that $y \neq \bot_L$ there exists an element x of L such that $x <_1 y$.

Let L be a lattice. The functor Join-IRR L yielding a subset of L is defined by:

(Def. 6) Join-IRR L = $\{a; a$ ranges over elements of L: $a \neq \bot_L \wedge \bigwedge_{b,c:\text{element of } L} (a = b \sqcup c \Rightarrow a = b \vee a = c)\}$.

One can prove the following three propositions:

(10) Let L be a lattice and x be an element of L. Then $x \in$ Join-IRR L if and only if the following conditions are satisfied:
(i) $x \neq \bot_L$, and
(ii) for all elements b, c of L such that $x = b \sqcup c$ holds $x = b$ or $x = c$.

(11) Let L be a finite distributive lattice and x be an element of L. Suppose $x \in$ Join-IRR L. Then there exists an element z of L such that $z < x$ and for every element y of L such that $y < x$ holds $y \leqslant z$.

(12) For every distributive finite lattice L and for every element x of L holds $\sup(\downarrow x \cap \text{Join-IRR}\, L) = x$.

3. Representation Theorem

Let P be a relational structure. The functor LOWER P yields a non empty set and is defined as follows:

(Def. 7) LOWER $P = \{X; X$ ranges over subsets of P: X is lower$\}$.

The following two propositions are true:

(13) Let L be a distributive finite lattice. Then there exists a map r from L into $\langle \text{LOWER}\,\text{sub}(\text{Join-IRR}\, L), \subseteq \rangle$ such that r is isomorphic and for every element a of L holds $r(a) = \downarrow a \cap \text{Join-IRR}\, L$.

(14) For every distributive finite lattice L holds L and $\langle \text{LOWER}\,\text{sub}(\text{Join-IRR}\, L), \subseteq \rangle$ are isomorphic.

Ring of sets is defined by:

(Def. 8) It includes lattice of it.

Let us note that there exists a ring of sets which is non empty.

Let X be a non empty ring of sets. One can verify that $\langle X, \subseteq \rangle$ is distributive and has l.u.b.'s and g.l.b.'s.

One can prove the following propositions:

(15) For every finite lattice L holds LOWER sub(Join-IRR L) is a ring of sets.

(16) Let L be a finite lattice. Then L is distributive if and only if there exists a non empty ring of sets X such that L and $\langle X, \subseteq \rangle$ are isomorphic.

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