FORMALIZED MATHEMATICS
Volume 9, Number 2, 2001
University of Białystok

On Segre's Product of Partial Line Spaces

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Summary. In this paper the concept of partial line spaces is presented. We also construct the Segre's product for a family of partial line spaces indexed by an arbitrary nonempty set.

MML Identifier: PENCIL_1.

The terminology and notation used in this paper have been introduced in the following articles: [16], [1], [2], [7], [14], [6], [13], [11], [9], [10], [8], [5], [17], [15], [12], [4], and [3].

1. Preliminaries

One can prove the following propositions:

(1) For all functions f, g such that $\prod f = \prod g$ holds if f is non-empty, then g is non-empty.

(2) For every set X holds $2 \subseteq \overline{\overline{X}}$ iff there exist sets x, y such that $x \in X$ and $y \in X$ and $x \neq y$.

(3) For every set X such that $2 \subseteq \overline{\overline{X}}$ and for every set x there exists a set y such that $y \in X$ and $x \neq y$.

(4) For every set X holds $2 \subseteq \overline{\overline{X}}$ iff X is non trivial.

(5) For every set X holds $3 \subseteq \overline{\overline{X}}$ iff there exist sets x, y, z such that $x \in X$ and $y \in X$ and $z \in X$ and $x \neq y$ and $x \neq z$ and $y \neq z$.

(6) For every set X such that $3 \subseteq \overline{\overline{X}}$ and for all sets x, y there exists a set z such that $z \in X$ and $x \neq z$ and $y \neq z$.

ISSN 1426–2630

2. Partial Line Spaces

Let S be a topological structure. A block of S is an element of the topology of S.

Let S be a topological structure and let x, y be points of S. We say that x, y are collinear if and only if:

(Def. 1) $x = y$ or there exists a block l of S such that $\{x, y\} \subseteq l$.

Let S be a topological structure and let T be a subset of the carrier of S. We say that T is closed under lines if and only if:

(Def. 2) For every block l of S such that $2 \subseteq \overline{\overline{l \cap T}}$ holds $l \subseteq T$.

We say that T is strong if and only if:

(Def. 3) For all points x, y of S such that $x \in T$ and $y \in T$ holds x, y are collinear.

Let S be a topological structure. We say that S is void if and only if:

(Def. 4) The topology of S is empty.

We say that S is degenerated if and only if:

(Def. 5) The carrier of S is a block of S.

We say that S has non trivial blocks if and only if:

(Def. 6) For every block k of S holds $2 \subseteq \overline{\overline{k}}$.

We say that S is identifying close blocks if and only if:

(Def. 7) For all blocks k, l of S such that $2 \subseteq \overline{\overline{k \cap l}}$ holds $k = l$.

We say that S is truly-partial if and only if:

(Def. 8) There exist points x, y of S such that x, y are not collinear.

We say that S has no isolated points if and only if:

(Def. 9) For every point x of S there exists a block l of S such that $x \in l$.

We say that S is connected if and only if the condition (Def. 10) is satisfied.

(Def. 10) Let x, y be points of S. Then there exists a finite sequence f of elements of the carrier of S such that
  - (i) $x = f(1)$,
  - (ii) $y = f(\operatorname{len} f)$, and
  - (iii) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ and for all points a, b of S such that $a = f(i)$ and $b = f(i+1)$ holds a, b are collinear.

We say that S is strongly connected if and only if the condition (Def. 11) is satisfied.

(Def. 11) Let x be a point of S and X be a subset of the carrier of S. Suppose X is closed under lines and strong. Then there exists a finite sequence f of elements of $2^{\text{the carrier of } S}$ such that
  - (i) $X = f(1)$,
  - (ii) $x \in f(\operatorname{len} f)$,

(iii) for every subset W of the carrier of S such that $W \in \operatorname{rng} f$ holds W is closed under lines and strong, and

(iv) for every natural number i such that $1 \leqslant i$ and $i < \operatorname{len} f$ holds $2 \subseteq \overline{\overline{f(i) \cap f(i+1)}}$.

One can prove the following propositions:

(7) Let X be a non empty set. Suppose $3 \subseteq \overline{\overline{X}}$. Let S be a topological structure. Suppose the carrier of $S = X$ and the topology of $S = \{L; L$ ranges over subsets of X: $2 = \overline{\overline{L}}\}$. Then S is non empty, non void, non degenerated, non truly-partial, and identifying close blocks and has non trivial blocks and no isolated points.

(8) Let X be a non empty set. Suppose $3 \subseteq \overline{\overline{X}}$. Let K be a subset of X. Suppose $\overline{\overline{K}} = 2$. Let S be a topological structure. Suppose the carrier of $S = X$ and the topology of $S = \{L; L$ ranges over subsets of X: $2 = \overline{\overline{L}}\} \setminus \{K\}$. Then S is non empty, non void, non degenerated, truly-partial, and identifying close blocks and has non trivial blocks and no isolated points.

One can verify that there exists a topological structure which is strict, non empty, non void, non degenerated, non truly-partial, and identifying close blocks and has non trivial blocks and no isolated points and there exists a topological structure which is strict, non empty, non void, non degenerated, truly-partial, and identifying close blocks and has non trivial blocks and no isolated points.

Let S be a non void topological structure. Note that the topology of S is non empty.

Let S be a topological structure with no isolated points and let x, y be points of S. Let us observe that x, y are collinear if and only if:

(Def. 12) There exists a block l of S such that $\{x, y\} \subseteq l$.

A PLS is a non empty non void non degenerated identifying close blocks topological structure with non trivial blocks.

Let F be a binary relation. We say that F is TopStruct-yielding if and only if:

(Def. 13) For every set x such that $x \in \operatorname{rng} F$ holds x is a topological structure.

Let us mention that every function which is TopStruct-yielding is also 1-sorted yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is TopStruct-yielding.

Let us note that there exists a function which is TopStruct-yielding.

Let F be a binary relation. We say that F is non-void-yielding if and only if:

(Def. 14) For every topological structure S such that $S \in \operatorname{rng} F$ holds S is non void.

Let F be a TopStruct-yielding function. Let us observe that F is non-void-yielding if and only if:

(Def. 15) For every set i such that $i \in \operatorname{rng} F$ holds i is a non void topological structure.

Let F be a binary relation. We say that F is trivial-yielding if and only if:

(Def. 16) For every set S such that $S \in \operatorname{rng} F$ holds S is trivial.

Let F be a binary relation. We say that F is non-Trivial-yielding if and only if:

(Def. 17) For every 1-sorted structure S such that $S \in \operatorname{rng} F$ holds S is non trivial.

Let us observe that every binary relation which is non-Trivial-yielding is also nonempty.

Let F be a 1-sorted yielding function. Let us observe that F is non-Trivial-yielding if and only if:

(Def. 18) For every set i such that $i \in \operatorname{rng} F$ holds i is a non trivial 1-sorted structure.

Let I be a non empty set, let A be a TopStruct-yielding many sorted set indexed by I, and let j be an element of I. Then $A(j)$ is a topological structure.

Let F be a binary relation. We say that F is PLS-yielding if and only if:

(Def. 19) For every set x such that $x \in \operatorname{rng} F$ holds x is a PLS.

One can verify the following observations:

* every function which is PLS-yielding is also nonempty and TopStruct-yielding,
* every TopStruct-yielding function which is PLS-yielding is also non-void-yielding, and
* every TopStruct-yielding function which is PLS-yielding is also non-Trivial-yielding.

Let I be a set. One can check that there exists a many sorted set indexed by I which is PLS-yielding.

Let I be a non empty set, let A be a PLS-yielding many sorted set indexed by I, and let j be an element of I. Then $A(j)$ is a PLS.

Let I be a set and let A be a many sorted set indexed by I. We say that A is Segre-like if and only if:

(Def. 20) There exists an element i of I such that for every element j of I such that $i \neq j$ holds $A(j)$ is non empty and trivial.

Let I be a set and let A be a many sorted set indexed by I. Note that $\{A\}$ is trivial-yielding.

The following proposition is true

(9) Let I be a non empty set, A be a many sorted set indexed by I, i be an

element of I, and S be a non trivial set. Then $A +\cdot (i, S)$ is non trivial-yielding.

Let I be a non empty set and let A be a many sorted set indexed by I. Observe that $\{A\}$ is Segre-like.

We now state two propositions:

(10) For every non empty set I and for every many sorted set A indexed by I and for all sets i, S holds $\{A\} +\cdot (i, S)$ is Segre-like.

(11) Let I be a non empty set, A be a nonempty 1-sorted yielding many sorted set indexed by I, and B be an element of the support of A. Then $\{B\}$ is a many sorted subset indexed by the support of A.

Let I be a non empty set and let A be a nonempty 1-sorted yielding many sorted set indexed by I. One can check that there exists a many sorted subset indexed by the support of A which is Segre-like, trivial-yielding, and non-empty.

Let I be a non empty set and let A be a non-Trivial-yielding 1-sorted yielding many sorted set indexed by I. Note that there exists a many sorted subset indexed by the support of A which is Segre-like, non trivial-yielding, and non-empty.

Let I be a non empty set. Observe that there exists a many sorted set indexed by I which is Segre-like and non trivial-yielding.

Let I be a non empty set and let B be a Segre-like non trivial-yielding many sorted set indexed by I. The functor index(B) yielding an element of I is defined by:

(Def. 21) $B(\mathrm{index}(B))$ is non trivial.

Next we state the proposition

(12) Let I be a non empty set, A be a Segre-like non trivial-yielding many sorted set indexed by I, and i be an element of I. If $i \neq \mathrm{index}(A)$, then $A(i)$ is non empty and trivial.

Let I be a non empty set. Note that every many sorted set indexed by I which is Segre-like and non trivial-yielding is also non-empty.

One can prove the following proposition

(13) Let I be a non empty set and A be a many sorted set indexed by I. Then $2 \subseteq \overline{\overline{\prod A}}$ if and only if A is non-empty and non trivial-yielding.

Let I be a non empty set and let B be a Segre-like non trivial-yielding many sorted set indexed by I. Note that $\prod B$ is non trivial.

3. Segre's Product

Let I be a non empty set and let A be a nonempty TopStruct-yielding many sorted set indexed by I. The functor Segre_Blocks A yields a family of subsets of $\prod$ (the support of A) and is defined by the condition (Def. 22).

(Def. 22) Let x be a set. Then $x \in$ Segre_Blocks A if and only if there exists a Segre-like many sorted subset B indexed by the support of A such that $x = \prod B$ and there exists an element i of I such that $B(i)$ is a block of $A(i)$.

Let I be a non empty set and let A be a nonempty TopStruct-yielding many sorted set indexed by I. The functor Segre_Product A yielding a non empty topological structure is defined as follows:

(Def. 23) Segre_Product $A = \langle \prod$ (the support of A),Segre_Blocks $A \rangle$.

The following propositions are true:

(14) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Then every point of Segre_Product A is a many sorted set indexed by I.

(15) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. If there exists an element i of I such that $A(i)$ is non void, then Segre_Product A is non void.

(16) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is non degenerated and there exists an element i of I such that $A(i)$ is non void. Then Segre_Product A is non degenerated.

(17) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ has non trivial blocks and there exists an element i of I such that $A(i)$ is non void. Then Segre_Product A has non trivial blocks.

(18) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is identifying close blocks and has non trivial blocks and there exists an element i of I such that $A(i)$ is non void. Then Segre_Product A is identifying close blocks.

Let I be a non empty set and let A be a PLS-yielding many sorted set indexed by I. Then Segre_Product A is a PLS.

One can prove the following propositions:

(19) Let T be a topological structure and S be a subset of the carrier of T. If S is trivial, then S is strong and closed under lines.

(20) Let S be an identifying close blocks topological structure, l be a block of S, and L be a subset of the carrier of S. If $L = l$, then L is closed under lines.

(21) Let S be a topological structure, l be a block of S, and L be a subset of the carrier of S. If $L = l$, then L is strong.

(22) For every non void topological structure S holds Ω_S is closed under lines.

(23) Let I be a non empty set, A be a Segre-like non trivial-yielding many sorted set indexed by I, and x, y be many sorted sets indexed by I. If $x \in \prod A$ and $y \in \prod A$, then for every set i such that $i \neq \mathrm{index}(A)$ holds $x(i) = y(i)$.

(24) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and x be a set. Then x is a block of Segre_Product A if and only if there exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that $x = \prod L$ and $L(\mathrm{index}(L))$ is a block of $A(\mathrm{index}(L))$.

(25) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and P be a many sorted set indexed by I. Suppose P is a point of Segre_Product A. Let i be an element of I and p be a point of $A(i)$. Then $P +\cdot (i, p)$ is a point of Segre_Product A.

(26) Let I be a non empty set and A, B be Segre-like non trivial-yielding many sorted sets indexed by I. Suppose $2 \subseteq \overline{\overline{\prod A \cap \prod B}}$. Then $\mathrm{index}(A) = \mathrm{index}(B)$ and for every set i such that $i \neq \mathrm{index}(A)$ holds $A(i) = B(i)$.

(27) Let I be a non empty set, A be a Segre-like non trivial-yielding many sorted set indexed by I, and N be a non trivial set. Then $A +\cdot (\mathrm{index}(A), N)$ is Segre-like and non trivial-yielding.

(28) Let S be a non empty non void identifying close blocks topological structure with no isolated points. If S is strongly connected, then S is connected.

(29) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and S be a subset of the carrier of Segre_Product A. Then S is non trivial, strong, and closed under lines if and only if there exists a Segre-like non trivial-yielding many sorted subset B indexed by the support of A such that $S = \prod B$ and for every subset C of the carrier of $A(\mathrm{index}(B))$ such that $C = B(\mathrm{index}(B))$ holds C is strong and closed under lines.

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Received May 29, 2000