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Weights of Continuous Lattices[1]

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Summary. This work is a continuation of formalization of [13]. Theorems from Chapter III, Section 4, pp. 170–171 are proved.

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The papers [25], [20], [1], [9], [12], [10], [22], [3], [15], [2], [23], [19], [26], [24], [27], [21], [8], [18], [5], [11], [6], [17], [16], [4], [14], and [7] provide the terminology and notation for this paper.

In this article we present several logical schemes. The scheme *UparrowUnion* deals with a relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let S be a family of subsets of the carrier of $\mathcal{A}$. If $S = \{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$, then $\uparrow\bigcup S = \bigcup\{\uparrow X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

The scheme *DownarrowUnion* deals with a relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let S be a family of subsets of the carrier of $\mathcal{A}$. If $S = \{X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$, then $\downarrow\bigcup S = \bigcup\{\downarrow X; X$ ranges over subsets of $\mathcal{A} : \mathcal{P}[X]\}$

for all values of the parameters.

Let L_1 be a lower-bounded continuous sup-semilattice and let B_1 be a CLbasis of L_1 with bottom. One can verify that $\langle \mathrm{Ids}(\mathrm{sub}(B_1)), \subseteq\rangle$ is algebraic.

Let L_1 be a continuous sup-semilattice. The functor $\mathrm{CLweight}\, L_1$ yields a cardinal number and is defined as follows:

(Def. 1) $\mathrm{CLweight}\, L_1 = \bigcap\{\overline{\overline{B_1}} : B_1$ ranges over CLbasis of L_1 with bottom$\}$.

We now state a number of propositions:

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(1) For every topological structure T and for every basis b of T holds weight $T \subseteq \overline{\overline{b}}$.

(2) For every topological structure T there exists a basis b of T such that $\overline{\overline{b}} = \text{weight}\, T$.

(3) For every continuous sup-semilattice L_1 and for every CLbasis B_1 of L_1 with bottom holds CLweight $L_1 \subseteq \overline{\overline{B_1}}$.

(4) For every continuous sup-semilattice L_1 there exists a CLbasis B_1 of L_1 with bottom such that $\overline{\overline{B_1}} = \text{CLweight}\, L_1$.

(5) For every algebraic lower-bounded lattice L_1 holds CLweight $L_1 = \overline{\overline{\text{the carrier of CompactSublatt}(L_1)}}$.

(6) Let T be a non empty topological space and L_1 be a continuous sup-semilattice. If $\langle$the topology of $T, \subseteq\rangle = L_1$, then every CLbasis of L_1 with bottom is a basis of T.

(7) Let T be a non empty topological space and L_1 be a continuous lower-bounded lattice. Suppose $\langle$the topology of $T, \subseteq\rangle = L_1$. Let B_1 be a basis of T and B_2 be a subset of L_1. If $B_1 = B_2$, then finsups(B_2) is a CLbasis of L_1 with bottom.

(8) Let T be a T_0 non empty topological space and L_1 be a continuous lower-bounded sup-semilattice. If $\langle$the topology of $T, \subseteq\rangle = L_1$, then if T is infinite, then weight $T = \text{CLweight}\, L_1$.

(9) Let T be a T_0 non empty topological space and L_1 be a continuous sup-semilattice. Suppose $\langle$the topology of $T, \subseteq\rangle = L_1$. Then $\overline{\overline{\text{the carrier of } T}} \subseteq \overline{\overline{\text{the carrier of } L_1}}$.

(10) For every T_0 non empty topological space T such that T is finite holds weight $T = \overline{\overline{\text{the carrier of } T}}$.

(11) Let T be a topological structure and L_1 be a continuous lower-bounded lattice. Suppose $\langle$the topology of $T, \subseteq\rangle = L_1$ and T is finite. Then CLweight $L_1 = \overline{\overline{\text{the carrier of } L_1}}$.

(12) Let L_1 be a continuous lower-bounded sup-semilattice, T_1 be a Scott topological augmentation of L_1, T_2 be a Lawson correct topological augmentation of L_1, and B_2 be a basis of T_2. Then $\{\uparrow V; V$ ranges over subsets of T_2: $V \in B_2\}$ is a basis of T_1.

(13) For all finite sets X, Y such that $X \subseteq Y$ and $\overline{\overline{X}} = \overline{\overline{Y}}$ holds $X = Y$.

(14) For every up-complete non empty poset L_1 such that L_1 is finite and for every element x of L_1 holds $x \in \text{compactbelow}(x)$.

(15) Every finite lattice is arithmetic.

One can check that every lattice which is finite is also arithmetic.

One can verify that there exists a relational structure which is trivial, reflexive, transitive, antisymmetric, lower-bounded, non empty, finite, and strict and has l.u.b.'s and g.l.b.'s.

One can prove the following proposition

(16) Let L_1 be a finite lattice and B_1 be a CLbasis of L_1 with bottom. Then $\overline{\overline{B_1}} = \mathrm{CLweight}\, L_1$ if and only if $B_1 =$ the carrier of CompactSublatt(L_1).

Let L_1 be a non empty reflexive relational structure, let A be a subset of the carrier of L_1, and let a be an element of L_1. The functor Way_Up(a, A) yields a subset of L_1 and is defined as follows:

(Def. 2) Way_Up$(a, A) = \twoheaduparrow a \setminus \uparrow A$.

Next we state a number of propositions:

(17) For every non empty reflexive relational structure L_1 and for every element a of L_1 holds Way_Up$(a, \emptyset_{(L_1)}) = \twoheaduparrow a$.

(18) For every non empty poset L_1 and for every subset A of L_1 and for every element a of L_1 such that $a \in \uparrow A$ holds Way_Up$(a, A) = \emptyset$.

(19) For every non empty finite reflexive transitive relational structure L_1 holds Ids(L_1) is finite.

(20) For every continuous lower-bounded sup-semilattice L_1 such that L_1 is infinite holds every CLbasis of L_1 with bottom is infinite.

(21) For every set d and for every finite sequence p and for every natural number i such that $i \in \mathrm{dom}\, p$ holds $(\langle d \rangle ^\frown p)(i+1) = p(i)$.

(22) For every finite sequence p and for every set x holds $(\langle x \rangle ^\frown p)_{\restriction 1} = p$.

(23) For every complete non empty poset L_1 and for every element x of L_1 such that x is compact holds $x = \inf \twoheaduparrow x$.

(24) Let L_1 be a continuous lower-bounded sup-semilattice. Suppose L_1 is infinite. Let B_1 be a CLbasis of L_1 with bottom. Then $\overline{\overline{\{\text{Way\_Up}(a, A); a \text{ ranges over elements of } L_1, A \text{ ranges over finite subsets of } L_1\colon a \in B_1 \wedge A \subseteq B_1\}}} \subseteq \overline{\overline{B_1}}$.

(25) For every Lawson complete top-lattice T and for every finite subset X of T holds $-\uparrow X$ is open and $-\downarrow X$ is open.

(26) Let L_1 be a continuous lower-bounded sup-semilattice, T be a Lawson correct topological augmentation of L_1, and B_1 be a CLbasis of L_1 with bottom. Then {Way_Up(a, A); a ranges over elements of L_1, A ranges over finite subsets of L_1: $a \in B_1 \wedge A \subseteq B_1$} is a basis of T.

(27) Let L_1 be a continuous lower-bounded sup-semilattice, T be a Scott topological augmentation of L_1, and b be a basis of T. Then $\{\twoheaduparrow \inf u; u$ ranges over subsets of T: $u \in b\}$ is a basis of T.

(28) Let L_1 be a continuous lower-bounded sup-semilattice, T be a Scott topological augmentation of L_1, and B_1 be a basis of T. If B_1 is infinite, then $\{\inf u; u$ ranges over subsets of T: $u \in B_1\}$ is infinite.

(29) Let L_1 be a continuous lower-bounded sup-semilattice and T be a Scott topological augmentation of L_1. Then CLweight L_1 = weight T.

(30) Let L_1 be a continuous lower-bounded sup-semilattice and T be a Lawson correct topological augmentation of L_1. Then CLweight L_1 = weight T.

(31) Let L_1, L_2 be non empty relational structures. Suppose L_1 and L_2 are isomorphic. Then $\overline{\overline{\text{the carrier of } L_1}} = \overline{\overline{\text{the carrier of } L_2}}$.

(32) Let L_1 be a continuous lower-bounded sup-semilattice and B_1 be a CLbasis of L_1 with bottom. If $\overline{\overline{B_1}}$ = CLweight L_1, then CLweight L_1 = CLweight$\langle \mathrm{Ids}(\mathrm{sub}(B_1)), \subseteq \rangle$.

Let L_1 be a continuous lower-bounded sup-semilattice. Note that $\langle \sigma(L_1), \subseteq \rangle$ is continuous and has l.u.b.'s.

Next we state two propositions:

(33) For every continuous lower-bounded sup-semilattice L_1 holds CLweight L_1 $\subseteq$ CLweight$\langle \sigma(L_1), \subseteq \rangle$.

(34) For every continuous lower-bounded sup-semilattice L_1 such that L_1 is infinite holds CLweight L_1 = CLweight$\langle \sigma(L_1), \subseteq \rangle$.

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