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# Circuit Generated by Terms and Circuit Calculating Terms

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**Summary.** In the paper we investigate the dependence between the structure of circuits and sets of terms. Circuits in our terminology (see [19]) are treated as locally-finite many sorted algebras over special signatures. Such approach enables to formalize every real circuit. The goal of this investigation is to specify circuits by terms and, enentualy, to have methods of formal verification of real circuits. The following notation is introduced in this paper:

- structural equivalence of circuits, i.e. equivalence of many sorted signatures,
- embedding of a circuit into another one,
- similarity of circuits (a concept narrower than isomorphism of many sorted algebras over equivalent signatures),
- calculation of terms by a circuit according to an algebra,
- specification of circuits by terms and an algebra.

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The articles [27], [3], [18], [19], [20], [11], [10], [17], [12], [13], [14], [22], [21], [9], [25], [1], [15], [24], [7], [28], [26], [23], [2], [5], [6], [8], [16], and [4] provide the terminology and notation for this paper.

## 1. Circuit Structure Generated by Terms

One can prove the following proposition

(1) Let $S$ be a non empty non void many sorted signature, $A$ be a non-empty algebra over $S$, $V$ be a variables family of $A$, $t$ be a term of $S$ over $V$, and $T$ be a term of $A$ over $V$. If $T = t$, then the sort of $T =$ the sort of $t$.




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Let $D$ be a non empty set and let $X$ be a subset of $D$. Then $\mathrm{id}_X$ is a function from $X$ into $D$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a non empty subset of $S$-Terms($V$). The functor $X$-CircuitStr yields a non empty strict many sorted signature and is defined by the condition (Def. 1).

(Def. 1) $X$-CircuitStr $= \langle$Subtrees($X$), [: the operation symbols of $S$, {the carrier of $S$} :]-Subtrees($X$), [: the operation symbols of $S$, {the carrier of $S$} :]-ImmediateSubtrees($X$), $\mathrm{id}_{[:\text{the operation symbols of } S,\ \{\text{the carrier of } S\}:]\text{-Subtrees}(X)}\rangle$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a non empty subset of $S$-Terms($V$). Observe that $X$-CircuitStr is unsplit.

In the sequel $S$ denotes a non empty non void many sorted signature, $V$ denotes a non-empty many sorted set indexed by the carrier of $S$, $A$ denotes a non-empty algebra over $S$, and $X$ denotes a non empty subset of $S$-Terms($V$).

The following propositions are true:

(2) $X$-CircuitStr is void if and only if for every element $t$ of $X$ holds $t$ is root and $t(\emptyset) \notin$ [: the operation symbols of $S$, {the carrier of $S$} :].

(3) $X$ is a set with a compound term of $S$ over $V$ iff $X$-CircuitStr is non void.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a set with a compound term of $S$ over $V$. One can check that $X$-CircuitStr is non void.

The following three propositions are true:

(4)(i) Every vertex of $X$-CircuitStr is a term of $S$ over $V$, and

(ii) for every set $s$ such that $s \in$ the operation symbols of $X$-CircuitStr holds $s$ is a compound term of $S$ over $V$.

(5) Let $t$ be a vertex of $X$-CircuitStr. Then $t \in$ the operation symbols of $X$-CircuitStr if and only if $t$ is a compound term of $S$ over $V$.

(6) Let $X$ be a set with a compound term of $S$ over $V$ and $g$ be a gate of $X$-CircuitStr. Then (the result sort of $X$-CircuitStr)($g$) $= g$ and the result sort of $g = g$.

Let us consider $S$, $V$, let $X$ be a set with a compound term of $S$ over $V$, and let $g$ be a gate of $X$-CircuitStr. Note that Arity($g$) is decorated tree yielding.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a non empty subset of $S$-Terms($V$). Note that every vertex of $X$-CircuitStr is finite, function-like, and relation-like.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a non empty subset of $S$-Terms($V$). One can verify that every vertex of $X$-CircuitStr is decorated tree-like.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a set with a compound term of $S$ over $V$. One can check that every gate of $X$-CircuitStr is finite, function-like, and relation-like.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a set with a compound term of $S$ over $V$. Note that every gate of $X$-CircuitStr is decorated tree-like.

Next we state the proposition

(7) Let $X_1$, $X_2$ be non empty subsets of $S$-Terms($V$). Then the arity of $X_1$-CircuitStr $\approx$ the arity of $X_2$-CircuitStr and the result sort of $X_1$-CircuitStr $\approx$ the result sort of $X_2$-CircuitStr.

Let $X$, $Y$ be constituted of decorated trees sets. Note that $X \cup Y$ is constituted of decorated trees.

One can prove the following propositions:

(8) For all constituted of decorated trees non empty sets $X_1$, $X_2$ holds Subtrees($X_1 \cup X_2$) = Subtrees($X_1$) $\cup$ Subtrees($X_2$).

(9) For all constituted of decorated trees non empty sets $X_1$, $X_2$ and for every set $C$ holds $C$-Subtrees($X_1 \cup X_2$) = ($C$-Subtrees($X_1$)) $\cup$ ($C$-Subtrees($X_2$)).

(10) Let $X_1$, $X_2$ be constituted of decorated trees non empty sets. If every element of $X_1$ is finite and every element of $X_2$ is finite, then for every set $C$ holds $C$-ImmediateSubtrees($X_1 \cup X_2$) = ($C$-ImmediateSubtrees($X_1$))+·($C$-ImmediateSubtrees($X_2$)).

(11) For all non empty subsets $X_1$, $X_2$ of $S$-Terms($V$) holds ($X_1 \cup X_2$)-CircuitStr = ($X_1$-CircuitStr)+·($X_2$-CircuitStr).

(12) Let $x$ be a set. Then $x \in$ InputVertices($X$-CircuitStr) if and only if the following conditions are satisfied:

(i) $x \in$ Subtrees($X$), and

(ii) there exists a sort symbol $s$ of $S$ and there exists an element $v$ of $V(s)$ such that $x$ = the root tree of $\langle v, s \rangle$.

(13) For every set $X$ with a compound term of $S$ over $V$ and for every gate $g$ of $X$-CircuitStr holds $g = g(\emptyset)$-tree(Arity($g$)).

## 2. Circuit Generated by Terms

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of

$S$-Terms($V$), let $v$ be a vertex of $X$-CircuitStr, and let $A$ be an algebra over $S$. The sort of $v$ w.r.t. $A$ is defined as follows:

(Def. 2) For every term $u$ of $S$ over $V$ such that $u = v$ holds the sort of $v$ w.r.t. $A =$ (the sorts of $A$)(the sort of $u$).

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), let $v$ be a vertex of $X$-CircuitStr, and let $A$ be a non-empty algebra over $S$. Note that the sort of $v$ w.r.t. $A$ is non empty.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $X$ be a non empty subset of $S$-Terms($V$). Let us assume that $X$ is a set with a compound term of $S$ over $V$. Let $o$ be a gate of $X$-CircuitStr and let $A$ be an algebra over $S$. The action of $o$ w.r.t $A$ is a function and is defined by the condition (Def. 3).

(Def. 3) Let $X'$ be a set with a compound term of $S$ over $V$. Suppose $X' = X$. Let $o'$ be a gate of $X'$-CircuitStr. Suppose $o' = o$. Then the action of $o$ w.r.t $A =$ (the characteristics of $A$)$(o'(\emptyset)_{\mathbf{1}})$.

The scheme *MSFuncEx* deals with a non empty set $\mathcal{A}$, non-empty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a many sorted function $f$ from $\mathcal{B}$ into $\mathcal{C}$ such that for every element $i$ of $\mathcal{A}$ and for every element $a$ of $\mathcal{B}(i)$ holds $\mathcal{P}[i, a, f(i)(a)]$

provided the following requirement is met:

- For every element $i$ of $\mathcal{A}$ and for every element $a$ of $\mathcal{B}(i)$ there exists an element $b$ of $\mathcal{C}(i)$ such that $\mathcal{P}[i, a, b]$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), and let $A$ be an algebra over $S$. The functor $X$-CircuitSorts($A$) yielding a many sorted set indexed by the carrier of $X$-CircuitStr is defined by:

(Def. 4) For every vertex $v$ of $X$-CircuitStr holds ($X$-CircuitSorts($A$))($v$) = the sort of $v$ w.r.t. $A$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), and let $A$ be a non-empty algebra over $S$. Note that $X$-CircuitSorts($A$) is non-empty.

We now state the proposition

(14) Let $X$ be a set with a compound term of $S$ over $V$, $g$ be a gate of $X$-CircuitStr, and $o$ be an operation symbol of $S$. If $g(\emptyset) = \langle o$, the carrier of $S\rangle$, then ($X$-CircuitSorts($A$)) $\cdot$ Arity($g$) = (the sorts of $A$) $\cdot$ Arity($o$).

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), and let $A$ be a non-empty algebra over

$S$. The functor $X$-CircuitCharact($A$) yields a many sorted function from $(X\text{-CircuitSorts}(A))^{\#} \cdot$ the arity of $X$-CircuitStr into $(X\text{-CircuitSorts}(A)) \cdot$ the result sort of $X$-CircuitStr and is defined by:

(Def. 5) For every gate $g$ of $X$-CircuitStr such that $g \in$ the operation symbols of $X$-CircuitStr holds $(X\text{-CircuitCharact}(A))(g) =$ the action of $g$ w.r.t $A$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), and let $A$ be a non-empty algebra over $S$. The functor $X$-Circuit($A$) yielding a non-empty strict algebra over $X$-CircuitStr is defined by:

(Def. 6) $X\text{-Circuit}(A) = \langle X\text{-CircuitSorts}(A), X\text{-CircuitCharact}(A)\rangle$.

Next we state four propositions:

(15) For every vertex $v$ of $X$-CircuitStr holds (the sorts of $X$-Circuit($A$))($v$) = the sort of $v$ w.r.t. $A$.

(16) Let $A$ be a locally-finite non-empty algebra over $S$, $X$ be a set with a compound term of $S$ over $V$, and $g$ be an operation symbol of $X$-CircuitStr. Then $\mathrm{Den}(g, X\text{-Circuit}(A)) =$ the action of $g$ w.r.t $A$.

(17) Let $A$ be a locally-finite non-empty algebra over $S$, $X$ be a set with a compound term of $S$ over $V$, $g$ be an operation symbol of $X$-CircuitStr, and $o$ be an operation symbol of $S$. If $g(\emptyset) = \langle o,$ the carrier of $S\rangle$, then $\mathrm{Den}(g, X\text{-Circuit}(A)) = \mathrm{Den}(o, A)$.

(18) Let $A$ be a locally-finite non-empty algebra over $S$ and $X$ be a non empty subset of $S$-Terms($V$). Then $X$-Circuit($A$) is locally-finite.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a set with a compound term of $S$ over $V$, and let $A$ be a locally-finite non-empty algebra over $S$. Note that $X$-Circuit($A$) is locally-finite.

The following two propositions are true:

(19) Let $S$ be a non empty non void many sorted signature, $V$ be a non-empty many sorted set indexed by the carrier of $S$, $X_1$, $X_2$ be sets with compound terms of $S$ over $V$, and $A$ be a non-empty algebra over $S$. Then $X_1\text{-Circuit}(A) \approx X_2\text{-Circuit}(A)$.

(20) Let $S$ be a non empty non void many sorted signature, $V$ be a non-empty many sorted set indexed by the carrier of $S$, $X_1$, $X_2$ be sets with compound terms of $S$ over $V$, and $A$ be a non-empty algebra over $S$. Then $(X_1 \cup X_2)\text{-Circuit}(A) = (X_1\text{-Circuit}(A)) {+}{\cdot} (X_2\text{-Circuit}(A))$.

## 3. Correctness of a Term Circuit

In the sequel $S$ is a non empty non void many sorted signature, $A$ is a non-empty locally-finite algebra over $S$, $V$ is a variables family of $A$, and $X$ is a set with a compound term of $S$ over $V$.

Let $S$ be a non empty non void many sorted signature, let $A$ be a non-empty algebra over $S$, let $V$ be a variables family of $A$, and let $t$ be a decorated tree. Let us assume that $t$ is a term of $S$ over $V$. Let $f$ be a many sorted function from $V$ into the sorts of $A$. The functor $[\![t]\!]_A(f)$ is defined by:

(Def. 7) There exists a term $t'$ of $A$ over $V$ such that $t' = t$ and $[\![t]\!]_A(f) = t' \mathbin{@} f$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a set with a compound term of $S$ over $V$, let $A$ be a non-empty locally-finite algebra over $S$, and let $s$ be a state of $X$-Circuit$(A)$. A many sorted function from $V$ into the sorts of $A$ is said to be a valuation compatible with $s$ if it satisfies the condition (Def. 8).

(Def. 8) Let $x$ be a vertex of $S$ and $v$ be an element of $V(x)$. If the root tree of $\langle v, x\rangle \in$ Subtrees$(X)$, then it$(x)(v) = s($the root tree of $\langle v, x\rangle)$.

Next we state the proposition

(21) Let $s$ be a state of $X$-Circuit$(A)$, $f$ be a valuation compatible with $s$, and $n$ be a natural number. Then $f$ is a valuation compatible with Following$(s, n)$.

Let $x$ be a set, let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, and let $p$ be a finite sequence of elements of $S$-Terms$(V)$. One can verify that $x$-tree$(p)$ is finite.

The following propositions are true:

(22) Let $s$ be a state of $X$-Circuit$(A)$, $f$ be a valuation compatible with $s$, and $t$ be a term of $S$ over $V$. If $t \in$ Subtrees$(X)$, then Following$(s, 1 +$ height dom $t)$ is stable at $t$ and (Following$(s, 1 +$ height dom $t))(t) = [\![t]\!]_A(f)$.

(23) Suppose that it is not true that there exists a term $t$ of $S$ over $V$ and there exists an operation symbol $o$ of $S$ such that $t \in$ Subtrees$(X)$ and $t(\emptyset) = \langle o$, the carrier of $S\rangle$ and Arity$(o) = \emptyset$. Let $s$ be a state of $X$-Circuit$(A)$, $f$ be a valuation compatible with $s$, and $t$ be a term of $S$ over $V$. If $t \in$ Subtrees$(X)$, then Following$(s$, height dom $t)$ is stable at $t$ and (Following$(s$, height dom $t))(t) = [\![t]\!]_A(f)$.

## 4. Circuit Similarity

Let $X$ be a set. One can verify that $\mathrm{id}_X$ is one-to-one.

Let $f$ be an one-to-one function. One can verify that $f^{-1}$ is one-to-one. Let $g$ be an one-to-one function. Note that $g \cdot f$ is one-to-one.

Let $S_1$, $S_2$ be non empty many sorted signatures and let $f$, $g$ be functions. We say that $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$ if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $f$ is one-to-one,

(ii) $g$ is one-to-one,

(iii) $f$ and $g$ form morphism between $S_1$ and $S_2$, and

(iv) $f^{-1}$ and $g^{-1}$ form morphism between $S_2$ and $S_1$.

One can prove the following propositions:

(24) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. Suppose $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$. Then the carrier of $S_2 = f^\circ$(the carrier of $S_1$) and the operation symbols of $S_2 = g^\circ$(the operation symbols of $S_1$).

(25) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. Suppose $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$. Then $\mathrm{rng}\, f =$ the carrier of $S_2$ and $\mathrm{rng}\, g =$ the operation symbols of $S_2$.

(26) Let $S$ be a non empty many sorted signature. Then $S$ and $S$ are equivalent w.r.t. $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$.

(27) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. Suppose $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$. Then $S_2$ and $S_1$ are equivalent w.r.t. $f^{-1}$ and $g^{-1}$.

(28) Let $S_1$, $S_2$, $S_3$ be non empty many sorted signatures and $f_1$, $g_1$, $f_2$, $g_2$ be functions. Suppose $S_1$ and $S_2$ are equivalent w.r.t. $f_1$ and $g_1$ and $S_2$ and $S_3$ are equivalent w.r.t. $f_2$ and $g_2$. Then $S_1$ and $S_3$ are equivalent w.r.t. $f_2 \cdot f_1$ and $g_2 \cdot g_1$.

(29) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. Suppose $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$. Then $f^\circ\, \mathrm{InputVertices}(S_1) = \mathrm{InputVertices}(S_2)$ and $f^\circ\, \mathrm{InnerVertices}(S_1) = \mathrm{InnerVertices}(S_2)$.

Let $S_1$, $S_2$ be non empty many sorted signatures. We say that $S_1$ and $S_2$ are equivalent if and only if:

(Def. 10) There exist one-to-one functions $f$, $g$ such that $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$.

Let us notice that the predicate $S_1$ and $S_2$ are equivalent is reflexive and symmetric.

One can prove the following proposition

(30) Let $S_1$, $S_2$, $S_3$ be non empty many sorted signatures. Suppose $S_1$ and $S_2$ are equivalent and $S_2$ and $S_3$ are equivalent. Then $S_1$ and $S_3$ are equivalent.

Let $S_1$, $S_2$ be non empty many sorted signatures and let $f$ be a function. We say that $f$ preserves inputs of $S_1$ in $S_2$ if and only if:

(Def. 11) $f^\circ \mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_2)$.

Next we state four propositions:

(31) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. Suppose $f$ and $g$ form morphism between $S_1$ and $S_2$. Let $v$ be a vertex of $S_1$. Then $f(v)$ is a vertex of $S_2$.

(32) Let $S_1$, $S_2$ be non empty non void many sorted signatures and $f$, $g$ be functions. Suppose $f$ and $g$ form morphism between $S_1$ and $S_2$. Let $v$ be a gate of $S_1$. Then $g(v)$ is a gate of $S_2$.

(33) Let $S_1$, $S_2$ be non empty many sorted signatures and $f$, $g$ be functions. If $f$ and $g$ form morphism between $S_1$ and $S_2$, then $f^\circ \mathrm{InnerVertices}(S_1) \subseteq \mathrm{InnerVertices}(S_2)$.

(34) Let $S_1$, $S_2$ be circuit-like non void non empty many sorted signatures and $f$, $g$ be functions. Suppose $f$ and $g$ form morphism between $S_1$ and $S_2$. Let $v_1$ be a vertex of $S_1$. Suppose $v_1 \in \mathrm{InnerVertices}(S_1)$. Let $v_2$ be a vertex of $S_2$. If $v_2 = f(v_1)$, then the action at $v_2 = g$(the action at $v_1$).

Let $S_1$, $S_2$ be non empty many sorted signatures, let $f$, $g$ be functions, let $C_1$ be a non-empty algebra over $S_1$, and let $C_2$ be a non-empty algebra over $S_2$. We say that $f$ and $g$ form embedding of $C_1$ into $C_2$ if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) $f$ is one-to-one,

(ii) $g$ is one-to-one,

(iii) $f$ and $g$ form morphism between $S_1$ and $S_2$,

(iv) the sorts of $C_1 =$ (the sorts of $C_2$) $\cdot f$, and

(v) the characteristics of $C_1 =$ (the characteristics of $C_2$) $\cdot g$.

The following propositions are true:

(35) Let $S$ be a non empty many sorted signature and $C$ be a non-empty algebra over $S$. Then $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$ form embedding of $C$ into $C$.

(36) Let $S_1$, $S_2$, $S_3$ be non empty many sorted signatures, $f_1$, $g_1$, $f_2$, $g_2$ be functions, $C_1$ be a non-empty algebra over $S_1$, $C_2$ be a non-empty algebra over $S_2$, and $C_3$ be a non-empty algebra over $S_3$. Suppose $f_1$ and $g_1$ form embedding of $C_1$ into $C_2$ and $f_2$ and $g_2$ form embedding of $C_2$ into $C_3$. Then $f_2 \cdot f_1$ and $g_2 \cdot g_1$ form embedding of $C_1$ into $C_3$.

Let $S_1$, $S_2$ be non empty many sorted signatures, let $f$, $g$ be functions, let $C_1$ be a non-empty algebra over $S_1$, and let $C_2$ be a non-empty algebra over $S_2$.

We say that $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$ if and only if:

(Def. 13) $f$ and $g$ form embedding of $C_1$ into $C_2$ and $f^{-1}$ and $g^{-1}$ form embedding of $C_2$ into $C_1$.

The following propositions are true:

(37) Let $S_1$, $S_2$ be non empty many sorted signatures, $f$, $g$ be functions, $C_1$ be a non-empty algebra over $S_1$, and $C_2$ be a non-empty algebra over $S_2$. Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Then $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$.

(38) Let $S_1$, $S_2$ be non empty many sorted signatures, $f$, $g$ be functions, $C_1$ be a non-empty algebra over $S_1$, and $C_2$ be a non-empty algebra over $S_2$. Then $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$ if and only if the following conditions are satisfied:
  (i) $S_1$ and $S_2$ are equivalent w.r.t. $f$ and $g$,
  (ii) the sorts of $C_1$ = (the sorts of $C_2$) $\cdot f$, and
  (iii) the characteristics of $C_1$ = (the characteristics of $C_2$) $\cdot g$.

(39) Let $S$ be a non empty many sorted signature and $C$ be a non-empty algebra over $S$. Then $C$ and $C$ are similar w.r.t. $\mathrm{id}_{\text{the carrier of } S}$ and $\mathrm{id}_{\text{the operation symbols of } S}$.

(40) Let $S_1$, $S_2$ be non empty many sorted signatures, $f$, $g$ be functions, $C_1$ be a non-empty algebra over $S_1$, and $C_2$ be a non-empty algebra over $S_2$. Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Then $C_2$ and $C_1$ are similar w.r.t. $f^{-1}$ and $g^{-1}$.

(41) Let $S_1$, $S_2$, $S_3$ be non empty many sorted signatures, $f_1$, $g_1$, $f_2$, $g_2$ be functions, $C_1$ be a non-empty algebra over $S_1$, $C_2$ be a non-empty algebra over $S_2$, and $C_3$ be a non-empty algebra over $S_3$. Suppose $C_1$ and $C_2$ are similar w.r.t. $f_1$ and $g_1$ and $C_2$ and $C_3$ are similar w.r.t. $f_2$ and $g_2$. Then $C_1$ and $C_3$ are similar w.r.t. $f_2 \cdot f_1$ and $g_2 \cdot g_1$.

Let $S_1$, $S_2$ be non empty many sorted signatures, let $C_1$ be a non-empty algebra over $S_1$, and let $C_2$ be a non-empty algebra over $S_2$. We say that $C_1$ and $C_2$ are similar if and only if:

(Def. 14) There exist functions $f$, $g$ such that $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$.

For simplicity, we use the following convention: $G_1$, $G_2$ denote circuit-like non void non empty many sorted signatures, $f$, $g$ denote functions, $C_1$ denotes a non-empty circuit of $G_1$, and $C_2$ denotes a non-empty circuit of $G_2$.

Next we state a number of propositions:

(42) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$. Then
  (i) $\mathrm{dom}\, f$ = the carrier of $G_1$,
  (ii) $\mathrm{rng}\, f \subseteq$ the carrier of $G_2$,
  (iii) $\mathrm{dom}\, g$ = the operation symbols of $G_1$, and

(iv) $\operatorname{rng} g \subseteq$ the operation symbols of $G_2$.

(43) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$. Let $o_1$ be a gate of $G_1$ and $o_2$ be a gate of $G_2$. If $o_2 = g(o_1)$, then $\mathrm{Den}(o_2, C_2) = \mathrm{Den}(o_1, C_1)$.

(44) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$. Let $o_1$ be a gate of $G_1$ and $o_2$ be a gate of $G_2$. Suppose $o_2 = g(o_1)$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. If $s_1 = s_2 \cdot f$, then $o_2$ depends-on-in $s_2 = o_1$ depends-on-in $s_1$.

(45) If $f$ and $g$ form embedding of $C_1$ into $C_2$, then for every state $s$ of $C_2$ holds $s \cdot f$ is a state of $C_1$.

(46) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$. Let $s_2$ be a state of $C_2$ and $s_1$ be a state of $C_1$. Suppose $s_1 = s_2 \cdot f$ and for every vertex $v$ of $G_1$ such that $v \in \mathrm{InputVertices}(G_1)$ holds $s_2$ is stable at $f(v)$. Then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(47) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$ and $f$ preserves inputs of $G_1$ in $G_2$. Let $s_2$ be a state of $C_2$ and $s_1$ be a state of $C_1$. If $s_1 = s_2 \cdot f$, then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(48) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$ and $f$ preserves inputs of $G_1$ in $G_2$. Let $s_2$ be a state of $C_2$ and $s_1$ be a state of $C_1$. If $s_1 = s_2 \cdot f$, then for every natural number $n$ holds $\mathrm{Following}(s_1, n) = \mathrm{Following}(s_2, n) \cdot f$.

(49) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$ and $f$ preserves inputs of $G_1$ in $G_2$. Let $s_2$ be a state of $C_2$ and $s_1$ be a state of $C_1$. If $s_1 = s_2 \cdot f$, then if $s_2$ is stable, then $s_1$ is stable.

(50) Suppose $f$ and $g$ form embedding of $C_1$ into $C_2$ and $f$ preserves inputs of $G_1$ in $G_2$. Let $s_2$ be a state of $C_2$ and $s_1$ be a state of $C_1$. Suppose $s_1 = s_2 \cdot f$. Let $v_1$ be a vertex of $G_1$. Then $s_1$ is stable at $v_1$ if and only if $s_2$ is stable at $f(v_1)$.

(51) If $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$, then for every state $s$ of $C_2$ holds $s \cdot f$ is a state of $C_1$.

(52) Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. Then $s_1 = s_2 \cdot f$ if and only if $s_2 = s_1 \cdot f^{-1}$.

(53) If $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$, then $f^\circ \mathrm{InputVertices}(G_1) = \mathrm{InputVertices}(G_2)$ and $f^\circ \mathrm{InnerVertices}(G_1) = \mathrm{InnerVertices}(G_2)$.

(54) If $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$, then $f$ preserves inputs of $G_1$ in $G_2$.

(55) Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. If $s_1 = s_2 \cdot f$, then $\mathrm{Following}(s_1) = \mathrm{Following}(s_2) \cdot f$.

(56) Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. If $s_1 = s_2 \cdot f$, then for every natural number $n$ holds $\mathrm{Following}(s_1, n) = \mathrm{Following}(s_2, n) \cdot f$.

(57) Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. If $s_1 = s_2 \cdot f$, then $s_1$ is stable iff $s_2$ is stable.

(58) Suppose $C_1$ and $C_2$ are similar w.r.t. $f$ and $g$. Let $s_1$ be a state of $C_1$ and $s_2$ be a state of $C_2$. Suppose $s_1 = s_2 \cdot f$. Let $v_1$ be a vertex of $G_1$. Then $s_1$ is stable at $v_1$ if and only if $s_2$ is stable at $f(v_1)$.

## 5. Term Specification

Let $S$ be a non empty non void many sorted signature, let $A$ be a non-empty algebra over $S$, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $X$ be a non empty subset of $S$-Terms($V$), let $G$ be a circuit-like non void non empty many sorted signature, and let $C$ be a non-empty circuit of $G$. We say that $C$ calculates $X$ in $A$ if and only if:

(Def. 15) There exist $f$, $g$ such that $f$ and $g$ form embedding of $X$-Circuit($A$) into $C$ and $f$ preserves inputs of $X$-CircuitStr in $G$.

We say that $X$ and $A$ specify $C$ if and only if:

(Def. 16) $C$ and $X$-Circuit($A$) are similar.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $A$ be a non-empty algebra over $S$, let $X$ be a non empty subset of $S$-Terms($V$), let $G$ be a circuit-like non void non empty many sorted signature, and let $C$ be a non-empty circuit of $G$. Let us assume that $C$ calculates $X$ in $A$. An one-to-one function is said to be a sort map from $X$ and $A$ into $C$ if:

(Def. 17) It preserves inputs of $X$-CircuitStr in $G$ and there exists $g$ such that it and $g$ form embedding of $X$-Circuit($A$) into $C$.

Let $S$ be a non empty non void many sorted signature, let $V$ be a non-empty many sorted set indexed by the carrier of $S$, let $A$ be a non-empty algebra over $S$, let $X$ be a non empty subset of $S$-Terms($V$), let $G$ be a circuit-like non void non empty many sorted signature, and let $C$ be a non-empty circuit of $G$. Let us assume that $C$ calculates $X$ in $A$. Let $f$ be a sort map from $X$ and $A$ into $C$. An one-to-one function is said to be an operation map from $X$ and $A$ into $C$ obeying $f$ if:

(Def. 18) $f$ and it form embedding of $X$-Circuit($A$) into $C$.

The following propositions are true:

(59) Let $G$ be a circuit-like non void non empty many sorted signature and $C$ be a non-empty circuit of $G$. If $X$ and $A$ specify $C$, then $C$ calculates $X$ in $A$.

(60) Let $G$ be a circuit-like non void non empty many sorted signature and $C$ be a non-empty circuit of $G$. Suppose $C$ calculates $X$ in $A$. Let $f$ be

a sort map from $X$ and $A$ into $C$ and $t$ be a term of $S$ over $V$. Suppose $t \in \mathrm{Subtrees}(X)$. Let $s$ be a state of $C$. Then

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t)$ is stable at $f(t)$, and

(ii) for every state $s'$ of $X$-Circuit$(A)$ such that $s' = s \cdot f$ and for every valuation $h$ compatible with $s'$ holds (Following$(s, 1+\mathrm{height}\,\mathrm{dom}\,t))(f(t)) = [\![t]\!]_A(h)$.

(61) Let $G$ be a circuit-like non void non empty many sorted signature and $C$ be a non-empty circuit of $G$. Suppose $C$ calculates $X$ in $A$. Let $t$ be a term of $S$ over $V$. Suppose $t \in \mathrm{Subtrees}(X)$. Then there exists a vertex $v$ of $G$ such that for every state $s$ of $C$ holds

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t)$ is stable at $v$, and

(ii) there exists a sort map $f$ from $X$ and $A$ into $C$ such that for every state $s'$ of $X$-Circuit$(A)$ such that $s' = s \cdot f$ and for every valuation $h$ compatible with $s'$ holds (Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t))(v) = [\![t]\!]_A(h)$.

(62) Let $G$ be a circuit-like non void non empty many sorted signature and $C$ be a non-empty circuit of $G$. Suppose $X$ and $A$ specify $C$. Let $f$ be a sort map from $X$ and $A$ into $C$, $s$ be a state of $C$, and $t$ be a term of $S$ over $V$. Suppose $t \in \mathrm{Subtrees}(X)$. Then

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t)$ is stable at $f(t)$, and

(ii) for every state $s'$ of $X$-Circuit$(A)$ such that $s' = s \cdot f$ and for every valuation $h$ compatible with $s'$ holds (Following$(s, 1+\mathrm{height}\,\mathrm{dom}\,t))(f(t)) = [\![t]\!]_A(h)$.

(63) Let $G$ be a circuit-like non void non empty many sorted signature and $C$ be a non-empty circuit of $G$. Suppose $X$ and $A$ specify $C$. Let $t$ be a term of $S$ over $V$. Suppose $t \in \mathrm{Subtrees}(X)$. Then there exists a vertex $v$ of $G$ such that for every state $s$ of $C$ holds

(i) Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t)$ is stable at $v$, and

(ii) there exists a sort map $f$ from $X$ and $A$ into $C$ such that for every state $s'$ of $X$-Circuit$(A)$ such that $s' = s \cdot f$ and for every valuation $h$ compatible with $s'$ holds (Following$(s, 1 + \mathrm{height}\,\mathrm{dom}\,t))(v) = [\![t]\!]_A(h)$.

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