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Trigonometric Form of Complex Numbers

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The articles [13], [1], [2], [8], [11], [15], [9], [3], [10], [12], [4], [18], [5], [16], [6], [19], [14], [17], and [7] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let F be an add-associative right zeroed right complementable left distributive non empty double loop structure and x be an element of the carrier of F. Then $0_F \cdot x = 0_F$.

(2) Let F be an add-associative right zeroed right complementable right distributive non empty double loop structure and x be an element of the carrier of F. Then $x \cdot 0_F = 0_F$.

The scheme *Regr without 0* concerns a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[1]$

provided the parameters meet the following conditions:

- There exists a non empty natural number k such that $\mathcal{P}[k]$, and
- For every non empty natural number k such that $k \neq 1$ and $\mathcal{P}[k]$ there exists a non empty natural number n such that $n < k$ and $\mathcal{P}[n]$.

One can prove the following propositions:

(3) For every element z of $\mathbb{C}$ holds $\Re(z) \geqslant -|z|$.

(4) For every element z of $\mathbb{C}$ holds $\Im(z) \geqslant -|z|$.

(5) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\Re(z) \geqslant -|z|$.

(6) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\Im(z) \geqslant -|z|$.

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(7) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $|z|^{\mathbf{2}} = \Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}$.

(8) For all real numbers x_1, x_2, y_1, y_2 such that $x_1 + x_2 i_{\mathbb{C}_{\rm F}} = y_1 + y_2 i_{\mathbb{C}_{\rm F}}$ holds $x_1 = y_1$ and $x_2 = y_2$.

(9) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $z = \Re(z) + \Im(z) i_{\mathbb{C}_{\rm F}}$.

(10) $0_{\mathbb{C}_{\rm F}} = 0 + 0 i_{\mathbb{C}_{\rm F}}$.

(11) $0_{\mathbb{C}_{\rm F}} =$ the zero of $\mathbb{C}_{\rm F}$.

(12) For every unital non empty groupoid L and for every element x of the carrier of L holds $\mathrm{power}_L(x, 1) = x$.

(13) For every unital non empty groupoid L and for every element x of the carrier of L holds $\mathrm{power}_L(x, 2) = x \cdot x$.

(14) Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and n be a natural number. If $n > 0$, then $\mathrm{power}_L(0_L, n) = 0_L$.

(15) Let L be an associative commutative unital non empty groupoid, x, y be elements of the carrier of L, and n be a natural number. Then $\mathrm{power}_L(x \cdot y, n) = \mathrm{power}_L(x, n) \cdot \mathrm{power}_L(y, n)$.

(16) For every real number x such that $x > 0$ and for every natural number n holds $\mathrm{power}_{\mathbb{C}_{\rm F}}(x + 0 i_{\mathbb{C}_{\rm F}}, n) = x^n + 0 i_{\mathbb{C}_{\rm F}}$.

(17) For every real number x and for every natural number n such that $x \geqslant 0$ and $n \neq 0$ holds $\sqrt[n]{x}^{\,n} = x$.

2. Sinus and Cosinus Properties

One can prove the following propositions:

(20)[1] $\pi + \frac{\pi}{2} = \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi + \frac{\pi}{2} = 2 \cdot \pi$ and $\frac{3}{2} \cdot \pi - \pi = \frac{\pi}{2}$.

(21) $0 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$ and $0 < \pi$ and $-\frac{\pi}{2} < \frac{\pi}{2}$ and $\pi < 2 \cdot \pi$ and $\frac{\pi}{2} < \frac{3}{2} \cdot \pi$ and $-\frac{\pi}{2} < 0$ and $0 < 2 \cdot \pi$ and $\pi < \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi < 2 \cdot \pi$ and $0 < \frac{3}{2} \cdot \pi$.

(22) For all real numbers a, b, c, x such that $x \in]a, c[$ holds $x \in]a, b[$ or $x = b$ or $x \in]b, c[$.

(23) For every real number x such that $x \in]0, \pi[$ holds $\sin(x) > 0$.

(24) For every real number x such that $x \in [0, \pi]$ holds $\sin(x) \geqslant 0$.

(25) For every real number x such that $x \in]\pi, 2 \cdot \pi[$ holds $\sin(x) < 0$.

(26) For every real number x such that $x \in [\pi, 2 \cdot \pi]$ holds $\sin(x) \leqslant 0$.

(27) For every real number x such that $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ holds $\cos(x) > 0$.

(28) For every real number x such that $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ holds $\cos(x) \geqslant 0$.

[1] The notation of π has been changed, previously 'Pai'. The propositions (18) and (19) have been removed.

(29) For every real number x such that $x \in]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$ holds $\cos(x) < 0$.

(30) For every real number x such that $x \in [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ holds $\cos(x) \leqslant 0$.

(31) For every real number x such that $x \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ holds $\cos(x) > 0$.

(32) For every real number x such that $x \in [\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ holds $\cos(x) \geqslant 0$.

(33) For every real number x such that $0 \leqslant x$ and $x < 2 \cdot \pi$ and $\sin x = 0$ holds $x = 0$ or $x = \pi$.

(34) For every real number x such that $0 \leqslant x$ and $x < 2 \cdot \pi$ and $\cos x = 0$ holds $x = \frac{\pi}{2}$ or $x = \frac{3}{2} \cdot \pi$.

(35) sin is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(36) sin is decreasing on $]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$.

(37) cos is decreasing on $]0, \pi[$.

(38) cos is increasing on $]\pi, 2 \cdot \pi[$.

(39) sin is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(40) sin is decreasing on $[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$.

(41) cos is decreasing on $[0, \pi]$.

(42) cos is increasing on $[\pi, 2 \cdot \pi]$.

(43) sin is continuous on $\mathbb{R}$ and for all real numbers x, y holds sin is continuous on $[x, y]$ and sin is continuous on $]x, y[$.

(44) cos is continuous on $\mathbb{R}$ and for all real numbers x, y holds cos is continuous on $[x, y]$ and cos is continuous on $]x, y[$.

(45) For every real number x holds $\sin(x) \in [-1, 1]$ and $\cos(x) \in [-1, 1]$.

(46) $\mathrm{rng}\sin = [-1, 1]$.

(47) $\mathrm{rng}\cos = [-1, 1]$.

(48) $\mathrm{rng}(\sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$.

(49) $\mathrm{rng}(\sin \restriction [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]) = [-1, 1]$.

(50) $\mathrm{rng}(\cos \restriction [0, \pi]) = [-1, 1]$.

(51) $\mathrm{rng}(\cos \restriction [\pi, 2 \cdot \pi]) = [-1, 1]$.

3. Argument of Complex Number

Let z be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $\mathrm{Arg}\, z$ yielding a real number is defined as follows:

(Def. 1)(i) $z = |z| \cdot \cos \mathrm{Arg}\, z + (|z| \cdot \sin \mathrm{Arg}\, z) i_{\mathbb{C}_{\mathrm{F}}}$ and $0 \leqslant \mathrm{Arg}\, z$ and $\mathrm{Arg}\, z < 2 \cdot \pi$ if $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$,

(ii) $\mathrm{Arg}\, z = 0$, otherwise.

One can prove the following propositions:

(52) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $0 \leqslant \operatorname{Arg} z$ and $\operatorname{Arg} z < 2 \cdot \pi$.

(53) For every real number x such that $x \geqslant 0$ holds $\operatorname{Arg} x + 0 i_{\mathbb{C}_{\rm F}} = 0$.

(54) For every real number x such that $x < 0$ holds $\operatorname{Arg} x + 0 i_{\mathbb{C}_{\rm F}} = \pi$.

(55) For every real number x such that $x > 0$ holds $\operatorname{Arg} 0 + x i_{\mathbb{C}_{\rm F}} = \frac{\pi}{2}$.

(56) For every real number x such that $x < 0$ holds $\operatorname{Arg} 0 + x i_{\mathbb{C}_{\rm F}} = \frac{3}{2} \cdot \pi$.

(57) $\operatorname{Arg} \mathbf{1}_{\mathbb{C}_{\rm F}} = 0$.

(58) $\operatorname{Arg} i_{\mathbb{C}_{\rm F}} = \frac{\pi}{2}$.

(59) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\operatorname{Arg} z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(60) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\operatorname{Arg} z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(61) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\operatorname{Arg} z \in]\pi, \frac{3}{2} \cdot \pi[$ iff $\Re(z) < 0$ and $\Im(z) < 0$.

(62) For every element z of the carrier of $\mathbb{C}_{\rm F}$ holds $\operatorname{Arg} z \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ iff $\Re(z) > 0$ and $\Im(z) < 0$.

(63) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Im(z) > 0$ holds $\sin \operatorname{Arg} z > 0$.

(64) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Im(z) < 0$ holds $\sin \operatorname{Arg} z < 0$.

(65) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Im(z) \geqslant 0$ holds $\sin \operatorname{Arg} z \geqslant 0$.

(66) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Im(z) \leqslant 0$ holds $\sin \operatorname{Arg} z \leqslant 0$.

(67) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Re(z) > 0$ holds $\cos \operatorname{Arg} z > 0$.

(68) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Re(z) < 0$ holds $\cos \operatorname{Arg} z < 0$.

(69) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Re(z) \geqslant 0$ holds $\cos \operatorname{Arg} z \geqslant 0$.

(70) For every element z of the carrier of $\mathbb{C}_{\rm F}$ such that $\Re(z) \leqslant 0$ and $z \neq 0_{\mathbb{C}_{\rm F}}$ holds $\cos \operatorname{Arg} z \leqslant 0$.

(71) For every real number x and for every natural number n holds $\operatorname{power}_{\mathbb{C}_{\rm F}}(\cos x + \sin x i_{\mathbb{C}_{\rm F}}, n) = \cos n \cdot x + \sin n \cdot x i_{\mathbb{C}_{\rm F}}$.

(72) Let z be an element of the carrier of $\mathbb{C}_{\rm F}$ and n be a natural number. If $z \neq 0_{\mathbb{C}_{\rm F}}$ or $n \neq 0$, then $\operatorname{power}_{\mathbb{C}_{\rm F}}(z, n) = |z|^n \cdot \cos n \cdot \operatorname{Arg} z + (|z|^n \cdot \sin n \cdot \operatorname{Arg} z) i_{\mathbb{C}_{\rm F}}$.

(73) For every real number x and for all natural numbers n, k such that $n \neq 0$ holds $\operatorname{power}_{\mathbb{C}_{\rm F}}(\cos \frac{x + 2 \cdot \pi \cdot k}{n} + \sin \frac{x + 2 \cdot \pi \cdot k}{n} i_{\mathbb{C}_{\rm F}}, n) = \cos x + \sin x i_{\mathbb{C}_{\rm F}}$.

(74) Let z be an element of the carrier of $\mathbb{C}_{\rm F}$ and n, k be natural numbers. If $n \neq 0$, then $z = \mathrm{power}_{\mathbb{C}_{\rm F}}(\sqrt[n]{|z|} \cdot \cos \frac{\mathrm{Arg}\, z + 2 \cdot \pi \cdot k}{n} + (\sqrt[n]{|z|} \cdot \sin \frac{\mathrm{Arg}\, z + 2 \cdot \pi \cdot k}{n}) i_{\mathbb{C}_{\rm F}}, n)$.

Let x be an element of the carrier of $\mathbb{C}_{\rm F}$ and let n be a non empty natural number. An element of $\mathbb{C}_{\rm F}$ is called a root of n, x if:

(Def. 2) $\mathrm{power}_{\mathbb{C}_{\rm F}}(\mathrm{it}, n) = x$.

We now state four propositions:

(75) Let x be an element of the carrier of $\mathbb{C}_{\rm F}$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|x|} \cdot \cos \frac{\mathrm{Arg}\, x + 2 \cdot \pi \cdot k}{n} + (\sqrt[n]{|x|} \cdot \sin \frac{\mathrm{Arg}\, x + 2 \cdot \pi \cdot k}{n}) i_{\mathbb{C}_{\rm F}}$ is a root of n, x.

(76) For every element x of the carrier of $\mathbb{C}_{\rm F}$ and for every root v of 1, x holds $v = x$.

(77) For every non empty natural number n and for every root v of n, $0_{\mathbb{C}_{\rm F}}$ holds $v = 0_{\mathbb{C}_{\rm F}}$.

(78) Let n be a non empty natural number, x be an element of the carrier of $\mathbb{C}_{\rm F}$, and v be a root of n, x. If $v = 0_{\mathbb{C}_{\rm F}}$, then $x = 0_{\mathbb{C}_{\rm F}}$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[5] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[8] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(**2**):265–269, 2001.
[9] Anna Justyna Milewska. The Hahn Banach theorem in the vector space over the field of complex numbers. *Formalized Mathematics*, 9(**2**):363–371, 2001.
[10] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[11] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[12] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[13] Wojciech Skaba and Michał Muzalewski. From double loops to fields. *Formalized Mathematics*, 2(**1**):185–191, 1991.
[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[15] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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