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Basic Properties of Extended Real Numbers

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Summary. We introduce product, quotient and absolute value, and we prove some basic properties of extended real numbers.

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The articles [3], [4], [5], [1], and [2] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x, y, z denote extended real numbers and a denotes a real number.

One can prove the following propositions:

(1) If $x \neq +\infty$ and $x \neq -\infty$, then x is a real number.

(2) $-\infty < +\infty$.

(3) If $x < y$, then $x \neq +\infty$ and $y \neq -\infty$.

(4) $x = +\infty$ iff $-x = -\infty$ and $x = -\infty$ iff $-x = +\infty$.

(5) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $x - -y = x + y$.

(6) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $x + -y = x - y$.

(7) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$, then $x \neq +\infty$ and $y \neq -\infty$.

(8) Suppose $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq -\infty$ but $y \neq -\infty$ or $z \neq +\infty$ and $x \neq +\infty$ or $z \neq -\infty$ but $x \neq -\infty$ or $z \neq +\infty$. Then $(x + y) + z = x + (y + z)$.

(9) If $-\infty < x$ and $x < +\infty$, then $x + -x = 0_{\overline{\mathbb{R}}}$ and $-x + x = 0_{\overline{\mathbb{R}}}$.

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(10) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $x - y = x + -y$.

(11) Suppose $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq +\infty$ but $y \neq -\infty$ or $z \neq -\infty$ and $x + y \neq +\infty$ or $y - z \neq -\infty$ but $x + y \neq -\infty$ or $y - z \neq +\infty$. Then $(x + y) - z = x + (y - z)$.

2. Operations of Multiplication, Quotient and Absolute Value on Extended Real Numbers

Let x, y be extended real numbers. The functor $x \cdot y$ yields an extended real number and is defined by the conditions (Def. 1).

(Def. 1)(i) There exist real numbers a, b such that $x = a$ and $y = b$ and $x \cdot y = a \cdot b$, or

(ii) $0_{\overline{\mathbb{R}}} < x$ and $y = +\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = +\infty$ or $x < 0_{\overline{\mathbb{R}}}$ and $y = -\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = -\infty$ but $x \cdot y = +\infty$, or

(iii) $x < 0_{\overline{\mathbb{R}}}$ and $y = +\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = +\infty$ or $0_{\overline{\mathbb{R}}} < x$ and $y = -\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = -\infty$ but $x \cdot y = -\infty$, or

(iv) $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$ but $x \cdot y = 0_{\overline{\mathbb{R}}}$.

The following propositions are true:

(12) Let x, y be extended real numbers. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x \cdot y = a \cdot b$, or

(ii) $0_{\overline{\mathbb{R}}} < x$ and $y = +\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = +\infty$ or $x < 0_{\overline{\mathbb{R}}}$ and $y = -\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = -\infty$ but $x \cdot y = +\infty$, or

(iii) $x < 0_{\overline{\mathbb{R}}}$ and $y = +\infty$ or $y < 0_{\overline{\mathbb{R}}}$ and $x = +\infty$ or $0_{\overline{\mathbb{R}}} < x$ and $y = -\infty$ or $0_{\overline{\mathbb{R}}} < y$ and $x = -\infty$ but $x \cdot y = -\infty$, or

(iv) $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$ but $x \cdot y = 0_{\overline{\mathbb{R}}}$.

(13) For all extended real numbers x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x \cdot y = a \cdot b$.

(14) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $+\infty \cdot x = +\infty$ and $x \cdot +\infty = +\infty$ and $-\infty \cdot x = -\infty$ and $x \cdot -\infty = -\infty$.

(15) For every extended real number x such that $x < 0_{\overline{\mathbb{R}}}$ holds $+\infty \cdot x = -\infty$ and $x \cdot +\infty = -\infty$ and $-\infty \cdot x = +\infty$ and $x \cdot -\infty = +\infty$.

(16) For all extended real numbers x, y such that $x = 0_{\overline{\mathbb{R}}}$ holds $x \cdot y = 0_{\overline{\mathbb{R}}}$ and $y \cdot x = 0_{\overline{\mathbb{R}}}$.

(17) For all extended real numbers x, y holds $x \cdot y = y \cdot x$.

Let x, y be extended real numbers. Let us notice that the functor $x \cdot y$ is commutative.

One can prove the following propositions:

(18) If $x = a$, then $0 < a$ iff $0_{\overline{\mathbb{R}}} < x$.

(19) If $x = a$, then $a < 0$ iff $x < 0_{\overline{\mathbb{R}}}$.

(20) If $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} < y$ or $x < 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$, then $0_{\overline{\mathbb{R}}} < x \cdot y$.

(21) If $0_{\overline{\mathbb{R}}} < x$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < y$, then $x \cdot y < 0_{\overline{\mathbb{R}}}$.

(22) $x \cdot y = 0_{\overline{\mathbb{R}}}$ iff $x = 0_{\overline{\mathbb{R}}}$ or $y = 0_{\overline{\mathbb{R}}}$.

(23) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(24) $-0_{\overline{\mathbb{R}}} = 0_{\overline{\mathbb{R}}}$.

(25) $0_{\overline{\mathbb{R}}} < x$ iff $-x < 0_{\overline{\mathbb{R}}}$ and $x < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} < -x$.

(26) $-x \cdot y = x \cdot -y$ and $-x \cdot y = (-x) \cdot y$.

(27) If $x \neq +\infty$ and $x \neq -\infty$ and $x \cdot y = +\infty$, then $y = +\infty$ or $y = -\infty$.

(28) If $x \neq +\infty$ and $x \neq -\infty$ and $x \cdot y = -\infty$, then $y = +\infty$ or $y = -\infty$.

(29) If $y \neq +\infty$ or $z \neq -\infty$ but $y \neq -\infty$ or $z \neq +\infty$ and $x \neq +\infty$ and $x \neq -\infty$, then $x \cdot (y + z) = x \cdot y + x \cdot z$.

(30) If $y \neq +\infty$ or $z \neq +\infty$ but $y \neq -\infty$ or $z \neq -\infty$ and $x \neq +\infty$ and $x \neq -\infty$, then $x \cdot (y - z) = x \cdot y - x \cdot z$.

Let x, y be extended real numbers. Let us assume that $x = -\infty$ or $x = +\infty$ but $y = -\infty$ or $y = +\infty$ but $y \neq 0_{\overline{\mathbb{R}}}$. The functor $\frac{x}{y}$ yielding an extended real number is defined by the conditions (Def. 2).

(Def. 2)(i) There exist real numbers a, b such that $x = a$ and $y = b$ and $\frac{x}{y} = \frac{a}{b}$, or

(ii) $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = +\infty$, or

(iii) $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = -\infty$, or

(iv) $y = -\infty$ or $y = +\infty$ but $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

The following four propositions are true:

(31) Let x, y be extended real numbers. Suppose $x = -\infty$ or $x = +\infty$ but $y = -\infty$ or $y = +\infty$ but $y \neq 0_{\overline{\mathbb{R}}}$. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $\frac{x}{y} = \frac{a}{b}$, or

(ii) $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = +\infty$, or

(iii) $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ or $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ but $\frac{x}{y} = -\infty$, or

(iv) $y = -\infty$ or $y = +\infty$ but $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

(32) Let x, y be extended real numbers. Suppose $y \neq 0_{\overline{\mathbb{R}}}$. Let a, b be real numbers. If $x = a$ and $y = b$, then $\frac{x}{y} = \frac{a}{b}$.

(33) For all extended real numbers x, y such that $x \neq -\infty$ but $x \neq +\infty$ but $y = -\infty$ or $y = +\infty$ holds $\frac{x}{y} = 0_{\overline{\mathbb{R}}}$.

(34) For every extended real number x such that $x \neq -\infty$ and $x \neq +\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$ holds $\frac{x}{x} = 1$.

Let x be an extended real number. The functor $|x|$ yielding an extended real number is defined as follows:

(Def. 3) $$|x| = \begin{cases} x, & \text{if } 0_{\overline{\mathbb{R}}} \leqslant x, \\ -x, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(35) For every extended real number x such that $0_{\overline{\mathbb{R}}} \leqslant x$ holds $|x| = x$.

(36) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $|x| = x$.

(37) For every extended real number x such that $x < 0_{\overline{\mathbb{R}}}$ holds $|x| = -x$.

(38) For all real numbers a, b holds $\overline{\mathbb{R}}(a \cdot b) = \overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b)$.

(39) For all real numbers a, b such that $b \neq 0$ holds $\overline{\mathbb{R}}(\frac{a}{b}) = \frac{\overline{\mathbb{R}}(a)}{\overline{\mathbb{R}}(b)}$.

(40) For all extended real numbers x, y such that $x \leqslant y$ and $x < +\infty$ and $-\infty < y$ holds $0_{\overline{\mathbb{R}}} \leqslant y - x$.

(41) For all extended real numbers x, y such that $x < y$ and $x < +\infty$ and $-\infty < y$ holds $0_{\overline{\mathbb{R}}} < y - x$.

(42) If $x \leqslant y$ and $0_{\overline{\mathbb{R}}} \leqslant z$, then $x \cdot z \leqslant y \cdot z$.

(43) If $x \leqslant y$ and $z \leqslant 0_{\overline{\mathbb{R}}}$, then $y \cdot z \leqslant x \cdot z$.

(44) If $x < y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $x \cdot z < y \cdot z$.

(45) If $x < y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $y \cdot z < x \cdot z$.

(46) Suppose x is a real number and y is a real number. Then $x < y$ if and only if there exist real numbers p, q such that $p = x$ and $q = y$ and $p < q$.

(47) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$ and $0_{\overline{\mathbb{R}}} < z$, then $\frac{x}{z} \leqslant \frac{y}{z}$.

(48) If $x \leqslant y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $\frac{x}{z} \leqslant \frac{y}{z}$.

(49) If $x \neq -\infty$ and $y \neq +\infty$ and $x \leqslant y$ and $z < 0_{\overline{\mathbb{R}}}$, then $\frac{y}{z} \leqslant \frac{x}{z}$.

(50) If $x \leqslant y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $\frac{y}{z} \leqslant \frac{x}{z}$.

(51) If $x < y$ and $0_{\overline{\mathbb{R}}} < z$ and $z \neq +\infty$, then $\frac{x}{z} < \frac{y}{z}$.

(52) If $x < y$ and $z < 0_{\overline{\mathbb{R}}}$ and $z \neq -\infty$, then $\frac{y}{z} < \frac{x}{z}$.

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