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Some Properties of Extended Real Numbers Operations: abs, min and max

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Summary. In this article, we extend some properties concerning real numbers to extended real numbers. Almost all properties included in this article are extended properties of other articles: [9], [6], [8], [10] and [7].

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The terminology and notation used in this paper are introduced in the following papers: [8], [4], [3], [5], [10], [11], [1], and [2].

1. Preliminaries

We follow the rules: x, y, w, z denote extended real numbers and a, b denote real numbers.

The following propositions are true:

(1) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $x + y = y + x$.

(2) If $x \neq +\infty$ and $x \neq -\infty$, then there exists y such that $x + y = 0_{\overline{\mathbb{R}}}$.

(3) If $x \neq +\infty$ and $x \neq -\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then there exists y such that $x \cdot y = \overline{1}$.

(4) $\overline{1} \cdot x = x$ and $x \cdot \overline{1} = x$ and $\overline{\mathbb{R}}(1) \cdot x = x$ and $x \cdot \overline{\mathbb{R}}(1) = x$.

(5) $0_{\overline{\mathbb{R}}} - x = -x$.

(6) $x \neq 0_{\overline{\mathbb{R}}}$ iff $-x \neq 0_{\overline{\mathbb{R}}}$.

(7) If $0_{\overline{\mathbb{R}}} \leqslant x$ and $0_{\overline{\mathbb{R}}} \leqslant y$, then $0_{\overline{\mathbb{R}}} \leqslant x + y$.

(8) If $0_{\overline{\mathbb{R}}} \leqslant x$ and $0_{\overline{\mathbb{R}}} < y$ or $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} \leqslant y$, then $0_{\overline{\mathbb{R}}} < x + y$.

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(9) If $x \leqslant 0_{\overline{\mathbb{R}}}$ and $y \leqslant 0_{\overline{\mathbb{R}}}$, then $x + y \leqslant 0_{\overline{\mathbb{R}}}$.

(10) If $x \leqslant 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $y \leqslant 0_{\overline{\mathbb{R}}}$, then $x + y < 0_{\overline{\mathbb{R}}}$.

(11) If $z \neq +\infty$ and $z \neq -\infty$ and $x + z = y$, then $x = y - z$.

(12) If $x \neq +\infty$ and $x \neq -\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $x \cdot \frac{\overline{1}}{x} = \overline{1}$ and $\frac{\overline{1}}{x} \cdot x = \overline{1}$.

(13) If $x \neq +\infty$ and $x \neq -\infty$, then $x - x = 0_{\overline{\mathbb{R}}}$.

(14) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $-(x + y) = -x + -y$ and $-(x + y) = -y - x$ and $-(x + y) = -x - y$.

(15) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $-(x - y) = -x + y$ and $-(x - y) = y - x$.

(16) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $-(-x + y) = x - y$ and $-(-x + y) = x + -y$.

(17) If $x = +\infty$ and $0_{\overline{\mathbb{R}}} < y$ and $y < +\infty$ or $x = -\infty$ and $y < 0_{\overline{\mathbb{R}}}$ and $-\infty < y$, then $\frac{x}{y} = +\infty$.

(18) If $x = +\infty$ and $y < 0_{\overline{\mathbb{R}}}$ and $-\infty < y$ or $x = -\infty$ and $0_{\overline{\mathbb{R}}} < y$ and $y < +\infty$, then $\frac{x}{y} = -\infty$.

(19) If $-\infty < y$ and $y < +\infty$ and $y \neq 0_{\overline{\mathbb{R}}}$, then $\frac{x \cdot y}{y} = x$ and $x \cdot \frac{y}{y} = x$.

(20) $\overline{1} < +\infty$ and $-\infty < \overline{1}$.

(21) If $x = +\infty$ or $x = -\infty$, then for every y such that $y \in \mathbb{R}$ holds $x + y \neq 0_{\overline{\mathbb{R}}}$.

(22) If $x = +\infty$ or $x = -\infty$, then for every y holds $x \cdot y \neq \overline{1}$.

(23) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $x + y < +\infty$, then $x \neq +\infty$ and $y \neq +\infty$.

(24) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $-\infty < x + y$, then $x \neq -\infty$ and $y \neq -\infty$.

(25) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $x - y < +\infty$, then $x \neq +\infty$ and $y \neq -\infty$.

(26) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $-\infty < x - y$, then $x \neq -\infty$ and $y \neq +\infty$.

(27) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $x + y < z$, then $x \neq +\infty$ and $y \neq +\infty$ and $z \neq -\infty$ and $x < z - y$.

(28) If $z \neq +\infty$ or $y \neq +\infty$ but $z \neq -\infty$ or $y \neq -\infty$ and $x < z - y$, then $x \neq +\infty$ and $y \neq +\infty$ and $z \neq -\infty$ and $x + y < z$.

(29) If $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$ and $x - y < z$, then $x \neq +\infty$ and $y \neq -\infty$ and $z \neq -\infty$ and $x < z + y$.

(30) If $z \neq +\infty$ or $y \neq -\infty$ but $z \neq -\infty$ or $y \neq +\infty$ and $x < z + y$, then $x \neq +\infty$ and $y \neq -\infty$ and $z \neq -\infty$ and $x - y < z$.

(31) If $x \neq +\infty$ or $y \neq -\infty$ and $x \neq -\infty$ or $y \neq +\infty$ and $y \neq +\infty$ or $z \neq +\infty$ and $y \neq -\infty$ or $z \neq -\infty$ and $x + y \leqslant z$, then $y \neq +\infty$ and $x \leqslant z - y$.

(32) If $x = +\infty$ and $y = -\infty$ and $x = -\infty$ and $y = +\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x \leqslant z - y$, then $y \neq +\infty$ and $x + y \leqslant z$.

(33) If $x \neq +\infty$ or $y \neq +\infty$ and $x \neq -\infty$ or $y \neq -\infty$ and $y \neq +\infty$ or $z \neq -\infty$ and $y \neq -\infty$ or $z \neq +\infty$ and $x - y \leqslant z$, then $y \neq -\infty$ and $x \leqslant z + y$.

(34) If $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = -\infty$ and $z = +\infty$ and $x \leqslant z + y$, then $y \neq -\infty$ and $x - y \leqslant z$.

(35) If $x \neq +\infty$ and $y \neq +\infty$, then $x + y \neq +\infty$.

(36) If $x \neq -\infty$ and $y \neq -\infty$, then $x + y \neq -\infty$.

(37) If $x \neq +\infty$ and $y \neq -\infty$, then $x - y \neq +\infty$.

(38) If $x \neq -\infty$ and $y \neq +\infty$, then $x - y \neq -\infty$.

(39) Suppose $x = +\infty$ and $y = -\infty$ and $x = -\infty$ and $y = +\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x = +\infty$ and $z = +\infty$ and $x = -\infty$ and $z = -\infty$. Then $(x + y) - z = x + (y - z)$.

(40) Suppose $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = +\infty$ and $z = -\infty$ and $y = -\infty$ and $z = +\infty$ and $x = +\infty$ and $z = +\infty$ and $x = -\infty$ and $z = -\infty$. Then $x - y - z = x - (y + z)$.

(41) Suppose $x = +\infty$ and $y = +\infty$ and $x = -\infty$ and $y = -\infty$ and $y = +\infty$ and $z = +\infty$ and $y = -\infty$ and $z = -\infty$ and $x = +\infty$ and $z = -\infty$ and $x = -\infty$ and $z = +\infty$. Then $(x - y) + z = x - (y - z)$.

(42) If $x \cdot y \neq +\infty$ and $x \cdot y \neq -\infty$, then x is a real number or y is a real number.

(43) $0_{\overline{\mathbb{R}}} < x$ and $0_{\overline{\mathbb{R}}} < y$ or $x < 0_{\overline{\mathbb{R}}}$ and $y < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} < x \cdot y$.

(44) $0_{\overline{\mathbb{R}}} < x$ and $y < 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < y$ iff $x \cdot y < 0_{\overline{\mathbb{R}}}$.

(45) $0_{\overline{\mathbb{R}}} \leqslant x$ or $0_{\overline{\mathbb{R}}} < x$ but $0_{\overline{\mathbb{R}}} \leqslant y$ or $0_{\overline{\mathbb{R}}} < y$ or $x \leqslant 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ but $y \leqslant 0_{\overline{\mathbb{R}}}$ or $y < 0_{\overline{\mathbb{R}}}$ iff $0_{\overline{\mathbb{R}}} \leqslant x \cdot y$.

(46) $x \leqslant 0_{\overline{\mathbb{R}}}$ or $x < 0_{\overline{\mathbb{R}}}$ but $0_{\overline{\mathbb{R}}} \leqslant y$ or $0_{\overline{\mathbb{R}}} < y$ or $0_{\overline{\mathbb{R}}} \leqslant x$ or $0_{\overline{\mathbb{R}}} < x$ but $y \leqslant 0_{\overline{\mathbb{R}}}$ or $y < 0_{\overline{\mathbb{R}}}$ iff $x \cdot y \leqslant 0_{\overline{\mathbb{R}}}$.

(47) $x \leqslant -y$ iff $y \leqslant -x$ and $-x \leqslant y$ iff $-y \leqslant x$.

(48) $x < -y$ iff $y < -x$ and $-x < y$ iff $-y < x$.

2. Basic Properties of abs for Extended Real Numbers

One can prove the following propositions:

(49) If $x = a$, then $|x| = |a|$.

(50) $|x| = x$ or $|x| = -x$.

(51) $0_{\overline{\mathbb{R}}} \leqslant |x|$.

(52) If $x \neq 0_{\overline{\mathbb{R}}}$, then $0_{\overline{\mathbb{R}}} < |x|$.

(53) $x = 0_{\overline{\mathbb{R}}}$ iff $|x| = 0_{\overline{\mathbb{R}}}$.

(54) If $|x| = -x$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $x < 0_{\overline{\mathbb{R}}}$.

(55) If $x \leqslant 0_{\overline{\mathbb{R}}}$, then $|x| = -x$.

(56) $|x \cdot y| = |x| \cdot |y|$.

(57) $-|x| \leqslant x$ and $x \leqslant |x|$.

(58) If $|x| < y$, then $-y < x$ and $x < y$.

(59) If $-y < x$ and $x < y$, then $0_{\overline{\mathbb{R}}} < y$ and $|x| < y$.

(60) $-y \leqslant x$ and $x \leqslant y$ iff $|x| \leqslant y$.

(61) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $|x+y| \leqslant |x|+|y|$.

(62) If $-\infty < x$ and $x < +\infty$ and $x \neq 0_{\overline{\mathbb{R}}}$, then $|x| \cdot |\frac{\overline{1}}{x}| = \overline{1}$.

(63) If $x = +\infty$ or $x = -\infty$, then $|x| \cdot |\frac{\overline{1}}{x}| = 0_{\overline{\mathbb{R}}}$.

(64) If $x \neq 0_{\overline{\mathbb{R}}}$, then $|\frac{\overline{1}}{x}| = \frac{\overline{1}}{|x|}$.

(65) If $x = -\infty$ or $x = +\infty$ and if $y = -\infty$ or $y = +\infty$ and if $y \neq 0_{\overline{\mathbb{R}}}$, then $|\frac{x}{y}| = \frac{|x|}{|y|}$.

(66) $|x| = |-x|$.

(67) If $x = +\infty$ or $x = -\infty$, then $|x| = +\infty$.

(68) If x is a real number or y is a real number, then $|x| - |y| \leqslant |x - y|$.

(69) If $x \neq +\infty$ or $y \neq +\infty$ and if $x \neq -\infty$ or $y \neq -\infty$, then $|x-y| \leqslant |x|+|y|$.

(70) $||x|| = |x|$.

(71) If $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ and $|x| \leqslant z$ and $|y| \leqslant w$, then $|x + y| \leqslant z + w$.

(72) If x is a real number or y is a real number, then $||x| - |y|| \leqslant |x - y|$.

(73) If $0_{\overline{\mathbb{R}}} \leqslant x \cdot y$, then $|x + y| = |x| + |y|$.

3. Definitions of min, max for Extended Real Numbers and their Basic Properties

Next we state the proposition

(74) If $x = a$ and $y = b$, then $b < a$ iff $y < x$ and $b \leqslant a$ iff $y \leqslant x$.

Let us consider x, y. The functor $\min(x, y)$ yielding an extended real number is defined by:

(Def. 1) $\min(x, y) = \begin{cases} x, & \text{if } x \leqslant y, \\ y, & \text{otherwise.} \end{cases}$

The functor $\max(x, y)$ yielding an extended real number is defined as follows:

(Def. 2) $\max(x, y) = \begin{cases} x, & \text{if } y \leqslant x, \\ y, & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(75) If $x = -\infty$ or $y = -\infty$, then $\min(x, y) = -\infty$.

(76) If $x = +\infty$ or $y = +\infty$, then $\max(x, y) = +\infty$.

(77) Let x, y be extended real numbers and a, b be real numbers. If $x = a$ and $y = b$, then $\min(x, y) = \min(a, b)$ and $\max(x, y) = \max(a, b)$.

(78) If $y \leqslant x$, then $\min(x, y) = y$.

(79) If $y \not\leqslant x$, then $\min(x, y) = x$.

(80) If $x \neq +\infty$ and $y \neq +\infty$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$, then $\min(x, y) = \frac{(x+y)-|x-y|}{\overline{\mathbb{R}}(2)}$.

(81) $\min(x, y) \leqslant x$ and $\min(y, x) \leqslant x$.

(82) $\min(x, x) = x$.

(83) $\min(x, y) = \min(y, x)$.

(84) $\min(x, y) = x$ or $\min(x, y) = y$.

(85) $x \leqslant y$ and $x \leqslant z$ iff $x \leqslant \min(y, z)$.

(86) If $\min(x, y) = x$, then $x \leqslant y$.

(87) If $\min(x, y) = y$, then $y \leqslant x$.

(88) $\min(x, \min(y, z)) = \min(\min(x, y), z)$.

(89) If $x \leqslant y$, then $\max(x, y) = y$.

(90) If $x \not\leqslant y$, then $\max(x, y) = x$.

(91) If $x \neq -\infty$ and $y \neq -\infty$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$, then $\max(x, y) = \frac{x+y+|x-y|}{\overline{\mathbb{R}}(2)}$.

(92) $x \leqslant \max(x, y)$ and $x \leqslant \max(y, x)$.

(93) $\max(x, x) = x$.

(94) $\max(x, y) = \max(y, x)$.

(95) $\max(x, y) = x$ or $\max(x, y) = y$.

(96) $y \leqslant x$ and $z \leqslant x$ iff $\max(y, z) \leqslant x$.

(97) If $\max(x, y) = x$, then $y \leqslant x$.

(98) If $\max(x, y) = y$, then $x \leqslant y$.

(99) $\max(x, \max(y, z)) = \max(\max(x, y), z)$.

(100) If $x \neq +\infty$ or $y \neq -\infty$ and if $x \neq -\infty$ or $y \neq +\infty$, then $\min(x, y) + \max(x, y) = x + y$.

(101) $\max(x, \min(x, y)) = x$ and $\max(\min(x, y), x) = x$ and $\max(\min(y, x), x) = x$ and $\max(x, \min(y, x)) = x$.

(102) $\min(x, \max(x, y)) = x$ and $\min(\max(x, y), x) = x$ and $\min(\max(y, x), x) = x$ and $\min(x, \max(y, x)) = x$.

(103) $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$ and $\min(\max(y, z), x) = \max(\min(y, x), \min(z, x))$.

(104) $\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$ and $\max(\min(y, z), x) = \min(\max(y, x), \max(z, x))$.

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