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The Incompleteness of the Lattice of Substitutions

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Summary. In [11] we proved that the lattice of substitutions, as defined in [9], is a Heyting lattice (i.e. it is pseudo-complemented and it has the zero element). We show that the lattice needs not to be complete. Obviously, the example has to be infinite, namely we can take the set of natural numbers as variables and a singleton as a set of constants. The incompleteness has been shown for lattices of substitutions defined in terms of [22] and relational structures [18].

MML Identifier: `HEYTING3`.

The terminology and notation used here are introduced in the following articles: [13], [20], [14], [4], [8], [17], [5], [10], [2], [22], [16], [1], [18], [6], [12], [21], [19], [9], [15], [3], and [7].

1. Preliminaries

The scheme *SSubsetUniq* deals with a relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

Let A_1, A_2 be subsets of $\mathcal{A}$. Suppose for every set x holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every set x holds $x \in A_2$ iff $\mathcal{P}[x]$. Then $A_1 = A_2$

for all values of the parameters.

Let A, x be sets. Observe that $[\!: A, \{x\} :\!]$ is function-like.

Next we state a number of propositions:

(1) For every odd natural number n holds $1 \leqslant n$.

(2) For every finite non empty subset X of $\mathbb{N}$ holds $\max X \in X$.

(3) For every finite non empty subset X of $\mathbb{N}$ there exists a natural number n such that $X \subseteq \operatorname{Seg} n \cup \{0\}$.

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(4) For every finite subset X of $\mathbb{N}$ there exists an odd natural number k such that $k \notin X$.

(5) Let k be a natural number and X be a finite non empty subset of $[\!: \mathbb{N}, \{k\} :\!]$. Then there exists a non empty natural number n such that $X \subseteq [\!: \mathrm{Seg}\, n \cup \{0\}, \{k\} :\!]$.

(6) Let m be a natural number and X be a finite non empty subset of $[\!: \mathbb{N}, \{m\} :\!]$. Then there exists a non empty natural number k such that $\langle 2 \cdot k+1, m \rangle \notin X$.

(7) Let m be a natural number and X be a finite subset of $[\!: \mathbb{N}, \{m\} :\!]$. Then there exists a natural number k such that for every natural number l such that $l \geqslant k$ holds $\langle l, m \rangle \notin X$.

(8) For every upper-bounded lattice L holds $\top_L = \top_{\mathrm{Poset}(L)}$.

(9) For every lower-bounded lattice L holds $\bot_L = \bot_{\mathrm{Poset}(L)}$.

(10) Let L be a lower-bounded non empty antisymmetric relational structure and a be an element of L. If $\bot_L \geqslant a$, then $a = \bot_L$.

2. Poset of Substitutions

Next we state four propositions:

(11) For every set V and for every finite set C and for all elements A, B of $\mathrm{Fin}(V \dot{\rightarrow} C)$ such that $A = \emptyset$ and $B \neq \emptyset$ holds $B \rightarrowtail A = \emptyset$.

(12) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds $\mathrm{SubstitutionSet}(V, C) \subseteq \mathrm{SubstitutionSet}(V', C')$.

(13) Let V, V', C, C' be sets, A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$, and B be an element of $\mathrm{Fin}(V' \dot{\rightarrow} C')$. If $V \subseteq V'$ and $C \subseteq C'$ and $A = B$, then $\mu A = \mu B$.

(14) Let V, V', C, C' be sets. Suppose $V \subseteq V'$ and $C \subseteq C'$. Then the join operation of $\mathrm{SubstLatt}(V, C)$ = (the join operation of $\mathrm{SubstLatt}(V', C')$)$\restriction[\!:$ the carrier of $\mathrm{SubstLatt}(V, C)$, the carrier of $\mathrm{SubstLatt}(V, C) :\!]$.

Let V, C be sets. The functor $\mathrm{SubstPoset}(V, C)$ yields a relational structure and is defined as follows:

(Def. 1) $\mathrm{SubstPoset}(V, C) = \mathrm{Poset}(\mathrm{SubstLatt}(V, C))$.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V, C)$ has l.u.b.'s and g.l.b.'s.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V, C)$ is reflexive antisymmetric and transitive.

One can prove the following propositions:

(15) Let V, C be sets and a, b be elements of $\mathrm{SubstPoset}(V, C)$. Then $a \leqslant b$ if and only if for every set x such that $x \in a$ there exists a set y such that $y \in b$ and $y \subseteq x$.

(16) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds SubstPoset(V, C) is a full relational substructure of SubstPoset(V', C').

Let n, k be natural numbers. The functor $\mathrm{PF_A}(n, k)$ yields an element of $\mathbb{N} \dot{\to} \{k\}$ and is defined as follows:

(Def. 2) For every set x holds $x \in \mathrm{PF_A}(n, k)$ iff there exists an odd natural number m such that $m \leqslant 2 \cdot n$ and $\langle m, k \rangle = x$ or $\langle 2 \cdot n, k \rangle = x$.

Let n, k be natural numbers. One can verify that $\mathrm{PF_A}(n, k)$ is finite.

Let n, k be natural numbers. The functor $\mathrm{PF_C}(n, k)$ yielding an element of $\mathbb{N} \dot{\to} \{k\}$ is defined by:

(Def. 3) For every set x holds $x \in \mathrm{PF_C}(n, k)$ iff there exists an odd natural number m such that $m \leqslant 2 \cdot n + 1$ and $\langle m, k \rangle = x$.

Let n, k be natural numbers. Note that $\mathrm{PF_C}(n, k)$ is finite.

The following four propositions are true:

(17) For all natural numbers n, k holds $\langle 2 \cdot n + 1, k \rangle \in \mathrm{PF_C}(n, k)$.

(18) For all natural numbers n, k holds $\mathrm{PF_C}(n, k) \cap \{\langle 2 \cdot n + 3, k \rangle\} = \emptyset$.

(19) For all natural numbers n, k holds $\mathrm{PF_C}(n+1, k) = \mathrm{PF_C}(n, k) \cup \{\langle 2 \cdot n + 3, k \rangle\}$.

(20) For all natural numbers n, k holds $\mathrm{PF_C}(n, k) \subset \mathrm{PF_C}(n + 1, k)$.

Let n, k be natural numbers. One can verify that $\mathrm{PF_A}(n, k)$ is non empty.

Next we state three propositions:

(21) For all natural numbers n, m, k holds $\mathrm{PF_A}(n, m) \not\subseteq \mathrm{PF_C}(k, m)$.

(22) For all natural numbers n, m, k such that $n \leqslant k$ holds $\mathrm{PF_C}(n, m) \subseteq \mathrm{PF_C}(k, m)$.

(23) For every natural number n holds $\mathrm{PF_A}(1, n) = \{\langle 1, n \rangle, \langle 2, n \rangle\}$.

Let n, k be natural numbers. The functor $\mathrm{PF_B}(n, k)$ yields an element of $\mathrm{Fin}(\mathbb{N} \dot{\to} \{k\})$ and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{PF_B}(n, k)$ iff there exists a non empty natural number m such that $m \leqslant n$ and $x = \mathrm{PF_A}(m, k)$ or $x = \mathrm{PF_C}(n, k)$.

The following propositions are true:

(24) For all natural numbers n, k and for every set x such that $x \in \mathrm{PF_B}(n + 1, k)$ there exists a set y such that $y \in \mathrm{PF_B}(n, k)$ and $y \subseteq x$.

(25) For all natural numbers n, k holds $\mathrm{PF_C}(n, k) \notin \mathrm{PF_B}(n + 1, k)$.

(26) For all natural numbers n, m, k such that $\mathrm{PF_A}(n, m) \subseteq \mathrm{PF_A}(k, m)$ holds $n = k$.

(27) For all natural numbers n, m, k holds $\mathrm{PF_C}(n, m) \subseteq \mathrm{PF_A}(k, m)$ iff $n < k$.

3. The Incompleteness

The following proposition is true

(28) For all natural numbers n, k holds $\mathrm{PF}_{\mathrm{B}}(n,k)$ is an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$.

Let k be a natural number. The functor $\mathrm{PF}_{\mathrm{D}}(k)$ yielding a subset of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ is defined as follows:

(Def. 5) For every set x holds $x \in \mathrm{PF}_{\mathrm{D}}(k)$ iff there exists a non empty natural number n such that $x = \mathrm{PF}_{\mathrm{B}}(n,k)$.

The following propositions are true:

(29) For every natural number k holds $\mathrm{PF}_{\mathrm{B}}(1,k) = \{\mathrm{PF}_{\mathrm{A}}(1,k), \mathrm{PF}_{\mathrm{C}}(1,k)\}$.

(30) For every natural number k holds $\mathrm{PF}_{\mathrm{B}}(1,k) \neq \{\emptyset\}$.

Let k be a natural number. Note that $\mathrm{PF}_{\mathrm{B}}(1,k)$ is non empty.

We now state four propositions:

(31) For all natural numbers n, k holds $\{\mathrm{PF}_{\mathrm{A}}(n,k)\}$ is an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$.

(32) Let k be a natural number, V, X be sets, and a be an element of $\mathrm{SubstPoset}(V,\{k\})$. If $X \in a$, then X is a finite subset of $[\!:V,\ \{k\}:\!]$.

(33) Let m be a natural number and a be an element of $\mathrm{SubstPoset}(\mathbb{N},\{m\})$. Suppose $\mathrm{PF}_{\mathrm{D}}(m) \geqslant a$. Let X be a non empty set. If $X \in a$, then it is not true that for every natural number n such that $\langle n,\ m\rangle \in X$ holds n is odd.

(34) Let k be a natural number, a, b be elements of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, and X be a subset of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$. If $a \leqslant X$ and $b \leqslant X$, then $a \sqcup b \leqslant X$.

Let k be a natural number. Note that there exists an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ which is non empty.

One can prove the following propositions:

(35) For every natural number n and for every element a of $\mathrm{SubstPoset}(\mathbb{N},\{n\})$ such that $\emptyset \in a$ holds $a = \{\emptyset\}$.

(36) Let k be a natural number and a be a non empty element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$. If $a \neq \{\emptyset\}$, then there exists a finite function f such that $f \in a$ and $f \neq \emptyset$.

(37) Let k be a natural number, a be a non empty element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, and a' be an element of $\mathrm{Fin}(\mathbb{N} \dot{\rightarrow} \{k\})$. If $a \neq \{\emptyset\}$ and $a = a'$, then $\mathrm{Involved}\, a'$ is a finite non empty subset of $\mathbb{N}$.

(38) Let k be a natural number, a be an element of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$, a' be an element of $\mathrm{Fin}(\mathbb{N} \dot{\rightarrow} \{k\})$, and B be a finite non empty subset of $\mathbb{N}$. Suppose $B = \mathrm{Involved}\, a'$ and $a' = a$. Let X be a set. If $X \in a$, then for every natural number l such that $l > \max B + 1$ holds $\langle l,\ k\rangle \notin X$.

(39) For every natural number k holds $\top_{\mathrm{SubstPoset}(\mathbb{N},\{k\})} = \{\emptyset\}$.

(40) For every natural number k holds $\perp_{\mathrm{SubstPoset}(\mathbb{N},\{k\})} = \emptyset$.

(41) For every natural number k and for all elements a, b of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ such that $a \leqslant b$ and $a = \{\emptyset\}$ holds $b = \{\emptyset\}$.

(42) For every natural number k and for all elements a, b of $\mathrm{SubstPoset}(\mathbb{N},\{k\})$ such that $a \leqslant b$ and $b = \emptyset$ holds $a = \emptyset$.

(43) For every natural number m and for every element a of $\mathrm{SubstPoset}(\mathbb{N},\{m\})$ such that $\mathrm{PF}_{\mathrm{D}}(m) \geqslant a$ holds $a \neq \{\emptyset\}$.

Let m be a natural number. One can verify that $\mathrm{SubstPoset}(\mathbb{N},\{m\})$ is non complete.

Let m be a natural number. One can check that $\mathrm{SubstLatt}(\mathbb{N},\{m\})$ is non complete.

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