FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

Some Lemmas for the Jordan Curve Theorem[1]

Andrzej Trybulec
University of Białystok

Summary. I present some miscellaneous simple facts that are still missing in the library. The only common feature is that, most of them, were needed as lemmas in the proof of the Jordan curve theorem.

MML Identifier: `JCT_MISC`.

The articles [11], [8], [17], [14], [9], [2], [3], [7], [1], [10], [4], [12], [5], [18], [19], [6], [15], [16], and [13] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *NonEmpty* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{A}\}$ is non empty

for all values of the parameters.

One can prove the following propositions:

(1) For all sets A, B, C such that $A \subseteq B$ and A misses C holds $A \subseteq B \setminus C$.

(2) For all sets X, Y such that X meets $\bigcup Y$ there exists a set Z such that $Z \in Y$ and X meets Z.

(3) For all sets A, B and for every function f such that $A \subseteq \operatorname{dom} f$ and $f^\circ A \subseteq B$ holds $A \subseteq f^{-1}(B)$.

(4) For every function f and for all sets A, B such that A misses B holds $f^{-1}(A)$ misses $f^{-1}(B)$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(5) Let S, X be sets, f be a function from S into X, and A be a subset of X such that if $X = \emptyset$, then $S = \emptyset$. Then $(f^{-1}(A))^{\rm c} = f^{-1}(A^{\rm c})$.

(6) Let S be a 1-sorted structure, X be a non empty set, f be a function from the carrier of S into X, and A be a subset of X. Then $-f^{-1}(A) = f^{-1}(A^{\rm c})$.

We use the following convention: i, j, m, n denote natural numbers and r, s, r_0, s_0, t denote real numbers.

Next we state several propositions:

(7) If $m \leqslant n$, then $n -' (n -' m) = m$.

(8) For every real number r such that $1 \leqslant r$ and $i \leqslant j$ holds $r^i \leqslant r^j$.

(9) For all real numbers a, b such that $r \in [a, b]$ and $s \in [a, b]$ holds $\frac{r+s}{2} \in [a, b]$.

(10) For every increasing sequence N_1 of naturals and for all i, j such that $i \leqslant j$ holds $N_1(i) \leqslant N_1(j)$.

(11) $||r_0 - s_0| - |r - s|| \leqslant |r_0 - r| + |s_0 - s|$.

(12) If $t \in]r, s[$, then $|t| < \max(|r|, |s|)$.

Let A, B, C be non empty sets and let f be a function from A into $[\!: B, C :\!]$. Then $\mathrm{pr1}(f)$ is a function from A into B and it can be characterized by the condition:

(Def. 1) For every element x of A holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

Then $\mathrm{pr2}(f)$ is a function from A into C and it can be characterized by the condition:

(Def. 2) For every element x of A holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

The scheme *DoubleChoice* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function a from $\mathcal{A}$ into $\mathcal{B}$ and there exists a function b from $\mathcal{A}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ holds $\mathcal{P}[i, a(i), b(i)]$

provided the parameters meet the following requirement:

- For every element i of $\mathcal{A}$ there exists an element a_1 of $\mathcal{B}$ and there exists an element b_1 of $\mathcal{C}$ such that $\mathcal{P}[i, a_1, b_1]$.

We now state the proposition

(13) Let S, T be non empty topological spaces and G be a subset of $[\!: S, T :\!]$. Suppose that for every point x of $[\!: S, T :\!]$ such that $x \in G$ there exists a subset G_1 of S and there exists a subset G_2 of T such that G_1 is open and G_2 is open and $x \in [\!: G_1, G_2 :\!]$ and $[\!: G_1, G_2 :\!] \subseteq G$. Then G is open.

2. Topological Properties of Sets of Real Numbers

One can prove the following proposition

(14) For all compact subsets A, B of $\mathbb{R}$ holds $A \cap B$ is compact.

Let A be a subset of $\mathbb{R}$. We say that A is connected if and only if:

(Def. 3) For all real numbers r, s such that $r \in A$ and $s \in A$ holds $[r, s] \subseteq A$.

The following proposition is true

(15) Let T be a non empty topological space, f be a continuous real map of T, and A be a subset of T. If A is connected, then $f^\circ A$ is connected.

Let A, B be subsets of $\mathbb{R}$. The functor $\rho(A, B)$ yielding a real number is defined by:

(Def. 4) There exists a subset X of $\mathbb{R}$ such that $X = \{|r - s|; r$ ranges over elements of $\mathbb{R}$, s ranges over elements of $\mathbb{R}$: $r \in A \wedge s \in B\}$ and $\rho(A, B) = \inf X$.

Let us notice that the functor $\rho(A, B)$ is commutative.

The following propositions are true:

(16) For all subsets A, B of $\mathbb{R}$ and for all r, s such that $r \in A$ and $s \in B$ holds $|r - s| \geqslant \rho(A, B)$.

(17) For all subsets A, B of $\mathbb{R}$ and for all non empty subsets C, D of $\mathbb{R}$ such that $C \subseteq A$ and $D \subseteq B$ holds $\rho(A, B) \leqslant \rho(C, D)$.

(18) For all non empty compact subsets A, B of $\mathbb{R}$ there exist real numbers r, s such that $r \in A$ and $s \in B$ and $\rho(A, B) = |r - s|$.

(19) For all non empty compact subsets A, B of $\mathbb{R}$ holds $\rho(A, B) \geqslant 0$.

(20) For all non empty compact subsets A, B of $\mathbb{R}$ such that A misses B holds $\rho(A, B) > 0$.

(21) Let e, f be real numbers and A, B be compact subsets of $\mathbb{R}$. Suppose A misses B and $A \subseteq [e, f]$ and $B \subseteq [e, f]$. Let S be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. Suppose that for every natural number i holds $S(i)$ is connected and $S(i)$ meets A and $S(i)$ meets B. Then there exists a real number r such that $r \in [e, f]$ and $r \notin A \cup B$ and for every natural number i there exists a natural number k such that $i \leqslant k$ and $r \in S(k)$.

References

[1] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[7] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[8] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[9] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[12] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[17] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received August 28, 2000