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Definitions and Basic Properties of Measurable Functions

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Summary. In this article we introduce some definitions concerning measurable functions and prove related properties.

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The papers [18], [10], [8], [9], [16], [6], [5], [2], [7], [1], [13], [12], [11], [19], [20], [14], [17], [3], [4], and [15] provide the notation and terminology for this paper.

1. Cardinal Numbers of $\mathbb{Z}$ and $\mathbb{Q}$

In this paper k is a natural number, r is a real number, i is an integer, and q is a rational number.

The subset $\mathbb{Z}_-$ of $\mathbb{R}$ is defined as follows:

(Def. 1) $r \in \mathbb{Z}_-$ iff there exists k such that $r = -k$.

Let us observe that $\mathbb{Z}_-$ is non empty.

Next we state three propositions:

(1) $\mathbb{N} \approx \mathbb{Z}_-$.

(2) $\mathbb{Z} = \mathbb{Z}_- \cup \mathbb{N}$.

(3) $\mathbb{N} \approx \mathbb{Z}$.

$\mathbb{Z}$ is a subset of $\mathbb{R}$.

Let n be a natural number. The functor $\mathbb{Q}(n)$ yields a subset of $\mathbb{Q}$ and is defined as follows:

(Def. 2) $q \in \mathbb{Q}(n)$ iff there exists i such that $q = \frac{i}{n}$.

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Let n be a natural number. Observe that $\mathbb{Q}(n+1)$ is non empty.

We now state two propositions:

(4) For every natural number n holds $\mathbb{Z} \approx \mathbb{Q}(n+1)$.

(5) $\mathbb{N} \approx \mathbb{Q}$.

2. Basic Operations of Extended Real Valued Functions

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let x be a set. Then $f(x)$ is an extended real number.

Let C be a non empty set and let f_1, f_2 be partial functions from C to $\overline{\mathbb{R}}$. The functor $f_1 + f_2$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 3) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \setminus ({f_1}^{-1}(\{-\infty\}) \cap {f_2}^{-1}(\{+\infty\}) \cup {f_1}^{-1}(\{+\infty\}) \cap {f_2}^{-1}(\{-\infty\}))$ and for every element c of C such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined by:

(Def. 4) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \setminus ({f_1}^{-1}(\{+\infty\}) \cap {f_2}^{-1}(\{+\infty\}) \cup {f_1}^{-1}(\{-\infty\}) \cap {f_2}^{-1}(\{-\infty\}))$ and for every element c of C such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

The functor $f_1\, f_2$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 5) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of C such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)(c) = f_1(c) \cdot f_2(c)$.

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let r be a real number. The functor $r\, f$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined as follows:

(Def. 6) $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)(c) = \overline{\mathbb{R}}(r) \cdot f(c)$.

The following proposition is true

(6) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. Suppose $r \neq 0$. Let c be an element of C. If $c \in \mathrm{dom}(r\, f)$, then $f(c) = \frac{(r\, f)(c)}{\overline{\mathbb{R}}(r)}$.

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $-f$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 7) $\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}(-f)$ holds $(-f)(c) = -f(c)$.

The extended real number $\overline{1}$ is defined by:

(Def. 8) $\overline{1} = 1$.

Let C be a non empty set, let f be a partial function from C to $\overline{\mathbb{R}}$, and let r be a real number. The functor $\frac{r}{f}$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 9) $\mathrm{dom}(\frac{r}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\overline{\mathbb{R}}}\})$ and for every element c of C such that $c \in \mathrm{dom}(\frac{r}{f})$ holds $(\frac{r}{f})(c) = \frac{\overline{\mathbb{R}}(r)}{f(c)}$.

One can prove the following proposition

(7) Let C be a non empty set and f be a partial function from C to $\overline{\mathbb{R}}$. Then $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\overline{\mathbb{R}}}\})$ and for every element c of C such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})(c) = \frac{\overline{1}}{f(c)}$.

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $|f|$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 10) $\mathrm{dom}\,|f| = \mathrm{dom}\, f$ and for every element c of C such that $c \in \mathrm{dom}\,|f|$ holds $|f|(c) = |f(c)|$.

We now state three propositions:

(8) For all extended real numbers x, y such that $x \neq +\infty$ or $y \neq -\infty$ but $x \neq -\infty$ or $y \neq +\infty$ holds $x + y = y + x$.

(9) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 + f_2 = f_2 + f_1$.

(10) For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1\, f_2 = f_2\, f_1$.

Let C be a non empty set and let f_1, f_2 be partial functions from C to $\overline{\mathbb{R}}$. Let us note that the functor $f_1 + f_2$ is commutative. Let us observe that the functor $f_1\, f_2$ is commutative.

3. Level Sets

Next we state several propositions:

(11) For every real number r there exists a natural number n such that $r \leqslant n$.

(12) For every real number r there exists a natural number n such that $-n \leqslant r$.

(13) For all real numbers r, s such that $r < s$ there exists a natural number n such that $\frac{1}{n+1} < s - r$.

(14) For all real numbers r, s such that for every natural number n holds $r - \frac{1}{n+1} \leqslant s$ holds $r \leqslant s$.

(15) For every extended real number a such that for every real number r holds $\overline{\mathbb{R}}(r) < a$ holds $a = +\infty$.

(16) For every extended real number a such that for every real number r holds $a < \overline{\mathbb{R}}(r)$ holds $a = -\infty$.

Let X be a set, let S be a σ-field of subsets of X, and let A be a set. We say that A is measurable on S if and only if:

(Def. 11) $A \in S$.

One can prove the following proposition

(17) Let X, A be sets and S be a σ-field of subsets of X. Then A is measurable on S if and only if for every σ-measure M on S holds A is measurable w.r.t. M.

For simplicity, we use the following convention: X is a non empty set, x is an element of X, f, g are partial functions from X to $\overline{\mathbb{R}}$, S is a σ-field of subsets of X, F is a function from $\mathbb{N}$ into S, A is a set, a is an extended real number, r, s are real numbers, and n is a natural number.

Let us consider X, f, a. The functor LE-dom(f,a) yielding a subset of X is defined by:

(Def. 12) $x \in$ LE-dom(f,a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $y < a$.

The functor LEQ-dom(f,a) yielding a subset of X is defined by:

(Def. 13) $x \in$ LEQ-dom(f,a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $y \leqslant a$.

The functor GT-dom(f,a) yields a subset of X and is defined as follows:

(Def. 14) $x \in$ GT-dom(f,a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a < y$.

The functor GTE-dom(f,a) yields a subset of X and is defined as follows:

(Def. 15) $x \in$ GTE-dom(f,a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a \leqslant y$.

The functor EQ-dom(f,a) yielding a subset of X is defined as follows:

(Def. 16) $x \in$ EQ-dom(f,a) iff $x \in \operatorname{dom} f$ and there exists an extended real number y such that $y = f(x)$ and $a = y$.

One can prove the following propositions:

(18) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap$ GTE-dom$(f,a) = A \setminus A \cap$ LE-dom(f,a).

(19) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap$ GT-dom$(f,a) = A \setminus A \cap$ LEQ-dom(f,a).

(20) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap$ LEQ-dom$(f,a) = A \setminus A \cap$ GT-dom(f,a).

(21) For all X, S, f, A, a such that $A \subseteq \operatorname{dom} f$ holds $A \cap$ LE-dom$(f,a) = A \setminus A \cap$ GTE-dom(f,a).

(22) For all X, S, f, A, a holds $A \cap$ EQ-dom$(f,a) = A \cap$ GTE-dom$(f,a) \cap$ LEQ-dom(f,a).

(23) For all X, S, F, f, A, r such that for every n holds $F(n) = A \cap$ GT-dom$(f, \overline{\mathbb{R}}(r - \frac{1}{n+1}))$ holds $A \cap$ GTE-dom$(f, \overline{\mathbb{R}}(r)) = \bigcap \operatorname{rng} F$.

(24) For all X, S, F, f, A and for every real number r such that for every n holds $F(n) = A \cap$ LE-dom$(f, \overline{\mathbb{R}}(r + \frac{1}{n+1}))$ holds $A \cap$ LEQ-dom$(f, \overline{\mathbb{R}}(r)) =$

$\bigcap \operatorname{rng} F$.

(25) For all X, S, F, f, A and for every real number r such that for every n holds $F(n) = A \cap \text{LEQ-dom}(f, \overline{\mathbb{R}}(r - \frac{1}{n+1}))$ holds $A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

(26) For all X, S, F, f, A, r such that for every n holds $F(n) = A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r + \frac{1}{n+1}))$ holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

(27) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(n))$ holds $A \cap \text{EQ-dom}(f, +\infty) = \bigcap \operatorname{rng} F$.

(28) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(n))$ holds $A \cap \text{LE-dom}(f, +\infty) = \bigcup \operatorname{rng} F$.

(29) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(-n))$ holds $A \cap \text{EQ-dom}(f, -\infty) = \bigcap \operatorname{rng} F$.

(30) For all X, S, F, f, A such that for every n holds $F(n) = A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(-n))$ holds $A \cap \text{GT-dom}(f, -\infty) = \bigcup \operatorname{rng} F$.

4. Measurable Functions

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\overline{\mathbb{R}}$, and let A be an element of S. We say that f is measurable on A if and only if:

(Def. 17) For every real number r holds $A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

In the sequel A, B are elements of S.

Next we state a number of propositions:

(31) Let given X, S, f, A. Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GTE-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(32) Let given X, S, f, A. Then f is measurable on A if and only if for every real number r holds $A \cap \text{LEQ-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(33) Let given X, S, f, A. Suppose $A \subseteq \operatorname{dom} f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r))$ is measurable on S.

(34) For all X, S, f, A, B such that $B \subseteq A$ and f is measurable on A holds f is measurable on B.

(35) For all X, S, f, A, B such that f is measurable on A and f is measurable on B holds f is measurable on $A \cup B$.

(36) For all X, S, f, A, r, s such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $A \cap \text{GT-dom}(f, \overline{\mathbb{R}}(r)) \cap \text{LE-dom}(f, \overline{\mathbb{R}}(s))$ is measurable on S.

(37) For all X, S, f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $A \cap \text{EQ-dom}(f, +\infty)$ is measurable on S.

(38) For all X, S, f, A such that f is measurable on A holds $A \cap$ EQ-dom$(f, -\infty)$ is measurable on S.

(39) For all X, S, f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $A \cap$ GT-dom$(f, -\infty) \cap$ LE-dom$(f, +\infty)$ is measurable on S.

(40) Let given X, S, f, g, A, r. Suppose f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$. Then $A \cap$ LE-dom$(f, \overline{\mathbb{R}}(r)) \cap$ GT-dom$(g, \overline{\mathbb{R}}(r))$ is measurable on S.

(41) For all X, S, f, A, r such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $r\,f$ is measurable on A.

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