FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

The Algebra of Polynomials[1]

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Summary. In this paper we define the algebra of formal power series and the algebra of polynomials over an arbitrary field and prove some properties of these structures. We also formulate and prove theorems showing some general properties of sequences. These preliminaries will be used for defining and considering linear functionals on the algebra of polynomials.

MML Identifier: `POLYALG1`.

The notation and terminology used here are introduced in the following papers: [9], [13], [1], [2], [3], [12], [8], [7], [11], [16], [5], [14], [10], [15], [6], and [4].

1. Preliminaries

Let F be a 1-sorted structure. We introduce algebra structures over F which are extensions of double loop structure and vector space structure over F and are systems

$\langle$ a carrier, an addition, a multiplication, a reverse-map, a zero, a unity, a left multiplication $\rangle$,

where the carrier is a set, the addition and the multiplication are binary operations on the carrier, the reverse-map is a unary operation on the carrier, the zero and the unity are elements of the carrier, and the left multiplication is a function from $[\!:$ the carrier of F, the carrier $:\!]$ into the carrier.

Let L be a non empty double loop structure. Note that there exists an algebra structure over L which is strict and non empty.

Let L be a non empty double loop structure and let A be a non empty algebra structure over L. We say that A is mix-associative if and only if:

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(Def. 1) For every element a of L and for all elements x, y of A holds $a \cdot (x \cdot y) = (a \cdot x) \cdot y$.

Let L be a non empty double loop structure. Note that there exists a non empty algebra structure over L which is well unital, distributive, vector space-like, and mix-associative.

Let L be a non empty double loop structure. An algebra of L is a well unital distributive vector space-like mix-associative non empty algebra structure over L.

Next we state two propositions:

(1) For all sets X, Y and for every function f from $[: X, Y :]$ into X holds $\mathrm{dom}\, f = [: X, Y :]$.

(2) For all sets X, Y and for every function f from $[: X, Y :]$ into Y holds $\mathrm{dom}\, f = [: X, Y :]$.

2. The Algebra of Formal Power Series

Let L be a non empty double loop structure. The functor Formal-Series L yields a strict non empty algebra structure over L and is defined by the conditions (Def. 2).

(Def. 2) For every set x holds $x \in$ the carrier of Formal-Series L iff x is a sequence of L and for all elements x, y of the carrier of Formal-Series L and for all sequences p, q of L such that $x = p$ and $y = q$ holds $x + y = p + q$ and for all elements x, y of the carrier of Formal-Series L and for all sequences p, q of L such that $x = p$ and $y = q$ holds $x \cdot y = p * q$ and for every element x of the carrier of Formal-Series L and for every sequence p of L such that $x = p$ holds $-x = -p$ and for every element a of L and for every element x of the carrier of Formal-Series L and for every sequence p of L such that $x = p$ holds $a \cdot x = a \cdot p$ and $0_{\text{Formal-Series}\, L} = \mathbf{0}.\, L$ and $\mathbf{1}_{\text{Formal-Series}\, L} = \mathbf{1}.\, L$.

Let L be an Abelian non empty double loop structure. Note that Formal-Series L is Abelian.

Let L be an add-associative non empty double loop structure. Note that Formal-Series L is add-associative.

Let L be a right zeroed non empty double loop structure. Note that Formal-Series L is right zeroed.

Let L be an add-associative right zeroed right complementable non empty double loop structure. Note that Formal-Series L is right complementable.

Let L be an Abelian add-associative right zeroed commutative non empty double loop structure. Observe that Formal-Series L is commutative.

Let L be an Abelian add-associative right zeroed right complementable unital associative distributive non empty double loop structure. Note that Formal-Series L is associative.

Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure. Note that Formal-Series L is right unital.

One can verify that there exists a non empty double loop structure which is add-associative, associative, right zeroed, left zeroed, right unital, left unital, right complementable, and distributive.

We now state three propositions:

(3) For every non empty set D and for every non empty finite sequence f of elements of D holds $f_{\lfloor 1} = f_{\restriction 1}$.

(4) For every non empty set D and for every non empty finite sequence f of elements of D holds $f = \langle f(1) \rangle \frown (f_{\restriction 1})$.

(5) Let L be an add-associative right zeroed left unital right complementable left distributive non empty double loop structure and p be a sequence of L. Then $\mathbf{1}.\, L * p = p$.

Let L be an add-associative right zeroed right complementable left unital left distributive non empty double loop structure. One can verify that Formal-Series L is left unital.

Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure. One can check that Formal-Series L is right distributive and Formal-Series L is left distributive.

We now state four propositions:

(6) Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, a be an element of L, and p, q be sequences of L. Then $a \cdot (p + q) = a \cdot p + a \cdot q$.

(7) Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, a, b be elements of L, and p be a sequence of L. Then $(a + b) \cdot p = a \cdot p + b \cdot p$.

(8) Let L be an associative non empty double loop structure, a, b be elements of L, and p be a sequence of L. Then $(a \cdot b) \cdot p = a \cdot (b \cdot p)$.

(9) Let L be an associative left unital non empty double loop structure and p be a sequence of L. Then (the unity of L) $\cdot\, p = p$.

Let L be an Abelian add-associative associative right zeroed right complementable left unital distributive non empty double loop structure. One can check that Formal-Series L is vector space-like.

We now state the proposition

(10) Let L be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure, a be

an element of L, and p, q be sequences of L. Then $a \cdot (p * q) = (a \cdot p) * q$.

Let L be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure. One can verify that Formal-Series L is mix-associative.

Let L be a left zeroed right zeroed add-associative left unital right unital right complementable distributive non empty double loop structure. Observe that Formal-Series L is well unital.

Let L be a 1-sorted structure and let A be an algebra structure over L. An algebra structure over L is said to be a subalgebra of A if it satisfies the conditions (Def. 3).

(Def. 3) The carrier of it $\subseteq$ the carrier of A and $\mathbf{1}_{\rm it} = \mathbf{1}_A$ and $0_{\rm it} = 0_A$ and the addition of it = (the addition of A)$\restriction$[: the carrier of it, the carrier of it :] and the multiplication of it = (the multiplication of A)$\restriction$[: the carrier of it, the carrier of it :] and the reverse-map of it = (the reverse-map of A)$\restriction$(the carrier of it) and the left multiplication of it = (the left multiplication of A)$\restriction$[: the carrier of L, the carrier of it :].

We now state four propositions:

(11) For every 1-sorted structure L holds every algebra structure A over L is a subalgebra of A.

(12) Let L be a 1-sorted structure and A, B, C be algebra structures over L. Suppose A is a subalgebra of B and B is a subalgebra of C. Then A is a subalgebra of C.

(13) Let L be a 1-sorted structure and A, B be algebra structures over L. Suppose A is a subalgebra of B and B is a subalgebra of A. Then the algebra structure of A = the algebra structure of B.

(14) Let L be a 1-sorted structure and A, B be algebra structures over L. Suppose the algebra structure of A = the algebra structure of B. Then A is a subalgebra of B and B is a subalgebra of A.

Let L be a non empty 1-sorted structure. Observe that there exists an algebra structure over L which is non empty and strict.

Let L be a 1-sorted structure and let B be an algebra structure over L. Observe that there exists a subalgebra of B which is strict.

Let L be a non empty 1-sorted structure and let B be a non empty algebra structure over L. Note that there exists a subalgebra of B which is strict and non empty.

Let L be a non empty groupoid, let B be a non empty algebra structure over L, and let A be a subset of B. We say that A is operations closed if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) A is linearly closed,

(ii) for all elements x, y of B such that $x \in A$ and $y \in A$ holds $x \cdot y \in A$,

(iii) for every element x of B such that $x \in A$ holds $-x \in A$,
(iv) $\mathbf{1}_B \in A$, and
(v) $0_B \in A$.

The following propositions are true:

(15) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x, y be elements of the carrier of B, and x', y' be elements of the carrier of A. If $x = x'$ and $y = y'$, then $x + y = x' + y'$.

(16) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x, y be elements of the carrier of B, and x', y' be elements of the carrier of A. If $x = x'$ and $y = y'$, then $x \cdot y = x' \cdot y'$.

(17) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, a be an element of the carrier of L, x be an element of the carrier of B, and x' be an element of the carrier of A. If $x = x'$, then $a \cdot x = a \cdot x'$.

(18) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subalgebra of B, x be an element of the carrier of B, and x' be an element of the carrier of A. If $x = x'$, then $-x = -x'$.

(19) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a non empty subalgebra of B. Then there exists a subset C of B such that the carrier of $A = C$ and C is operations closed.

(20) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a subset of B. Suppose A is operations closed. Then there exists a strict subalgebra C of B such that the carrier of $C = A$.

(21) Let L be a non empty groupoid, B be a non empty algebra structure over L, A be a non empty subset of B, and X be a family of subsets of the carrier of B. Suppose that for every set Y holds $Y \in X$ iff $Y \in 2^{\text{the carrier of } B}$ and there exists a subalgebra C of B such that $Y =$ the carrier of C and $A \subseteq Y$. Then $\bigcap X$ is operations closed.

Let L be a non empty groupoid, let B be a non empty algebra structure over L, and let A be a non empty subset of B. The functor GenAlg A yielding a strict non empty subalgebra of B is defined by the conditions (Def. 5).

(Def. 5)(i) $A \subseteq$ the carrier of GenAlg A, and
(ii) for every subalgebra C of B such that $A \subseteq$ the carrier of C holds the carrier of GenAlg $A \subseteq$ the carrier of C.

We now state the proposition

(22) Let L be a non empty groupoid, B be a non empty algebra structure over L, and A be a non empty subset of B. If A is operations closed, then the carrier of GenAlg $A = A$.

3. The Algebra of Polynomials

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. The functor Polynom-Algebra L yields a strict non empty algebra structure over L and is defined as follows:

(Def. 6) There exists a non empty subset A of Formal-Series L such that $A =$ the carrier of Polynom-Ring L and Polynom-Algebra $L =$ GenAlg A.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. One can verify that Polynom-Ring L is looplike.

The following propositions are true:

(23) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and A be a non empty subset of Formal-Series L. If $A =$ the carrier of Polynom-Ring L, then A is operations closed.

(24) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure. Then the double loop structure of Polynom-Algebra $L =$ Polynom-Ring L.

Acknowledgments

I would like to thank Dr Andrzej Trybulec for his helpful advices on this paper.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.
[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[7] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(**3**):461–470, 2001.
[8] Robert Milewski. The ring of polynomials. *Formalized Mathematics*, 9(**2**):339–346, 2001.
[9] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Formalized Mathematics*, 1(**5**):833–840, 1990.
[10] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[13] Wojciech Skaba and Michał Muzalewski. From double loops to fields. *Formalized Mathematics*, 2(**1**):185–191, 1991.

[14] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 24, 2001