FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

Fundamental Theorem of Algebra[1]

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MML Identifier: POLYNOM5.

The papers [18], [22], [19], [4], [16], [5], [12], [1], [3], [26], [24], [6], [7], [25], [13], [2], [20], [15], [14], [21], [9], [29], [27], [8], [10], [23], [28], [11], and [17] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For all natural numbers n, m such that $n \neq 0$ and $m \neq 0$ holds $(n \cdot m - n - m) + 1 \geqslant 0$.

(2) For all real numbers x, y such that $y > 0$ holds $\frac{\min(x,y)}{\max(x,y)} \leqslant 1$.

(3) For all real numbers x, y such that for every real number c such that $c > 0$ and $c < 1$ holds $c \cdot x \geqslant y$ holds $y \leqslant 0$.

(4) Let p be a finite sequence of elements of $\mathbb{R}$. Suppose that for every natural number n such that $n \in \operatorname{dom} p$ holds $p(n) \geqslant 0$. Let i be a natural number. If $i \in \operatorname{dom} p$, then $\sum p \geqslant p(i)$.

(5) For all real numbers x, y holds $-(x + y i_{\mathbb{C}_{\rm F}}) = -x + (-y) i_{\mathbb{C}_{\rm F}}$.

(6) For all real numbers x_1, y_1, x_2, y_2 holds $(x_1 + y_1 i_{\mathbb{C}_{\rm F}}) - (x_2 + y_2 i_{\mathbb{C}_{\rm F}}) = (x_1 - x_2) + (y_1 - y_2) i_{\mathbb{C}_{\rm F}}$.

(7) Let L be a commutative associative left unital distributive field-like non empty double loop structure and f, g, h be elements of the carrier of L. If $h \neq 0_L$, then if $h \cdot g = h \cdot f$ or $g \cdot h = f \cdot h$, then $g = f$.

[1]This work has been partially supported by TYPES grant IST-1999-29001.

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ISSN 1426–2630

In this article we present several logical schemes. The scheme *ExDHGrStrSeq* deals with a non empty groupoid $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of the carrier of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExDdoubleLoopStrSeq* deals with a non empty double loop structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of the carrier of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Next we state the proposition

(8) For every element z of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$ and for every natural number n holds $|\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(z, n)| = |z|^n$.

Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$. The functor $|p|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 1) $\mathrm{len}\,|p| = \mathrm{len}\,p$ and for every natural number n such that $n \in \mathrm{dom}\,p$ holds $|p|_n = |p_n|$.

We now state several propositions:

(9) $|\varepsilon_{(\text{the carrier of } \mathbb{C}_{\mathrm{F}})}| = \varepsilon_{\mathbb{R}}$.

(10) For every element x of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x \rangle| = \langle |x| \rangle$.

(11) For all elements x, y of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x, y \rangle| = \langle |x|, |y| \rangle$.

(12) For all elements x, y, z of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\langle x, y, z \rangle| = \langle |x|, |y|, |z| \rangle$.

(13) For all finite sequences p, q of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|p ^\frown q| = |p| ^\frown |q|$.

(14) Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ and x be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Then $|p ^\frown \langle x \rangle| = |p| ^\frown \langle |x| \rangle$ and $|\langle x \rangle ^\frown p| = \langle |x| \rangle ^\frown |p|$.

(15) For every finite sequence p of elements of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $|\sum p| \leqslant \sum |p|$.

2. Operations on Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a Polynomial of L, and let n be a natural number. The functor p^n yields a sequence of L and is defined by:

(Def. 2) $p^n = \mathrm{power}_{\text{Polynom-Ring}\,L}(p, n)$.

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a Polynomial of L, and let n be a natural number. One can verify that p^n is finite-Support.

One can prove the following propositions:

(16) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^0 = \mathbf{1}.\,L$.

(17) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^1 = p$.

(18) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^2 = p * p$.

(19) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p^3 = p * p * p$.

(20) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, p be a Polynomial of L, and n be a natural number. Then $p^{n+1} = p^n * p$.

(21) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{0}.\,L)^{n+1} = \mathbf{0}.\,L$.

(22) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{1}.\,L)^n = \mathbf{1}.\,L$.

(23) Let L be a field, p be a Polynomial of L, x be an element of the carrier of L, and n be a natural number. Then $\mathrm{eval}(p^n, x) = \mathrm{power}_L(\mathrm{eval}(p, x), n)$.

(24) Let L be a field and p be a Polynomial of L. If $\mathrm{len}\, p \neq 0$, then for every natural number n holds $\mathrm{len}(p^n) = (n \cdot \mathrm{len}\, p - n) + 1$.

Let L be a non empty groupoid, let p be a sequence of L, and let v be an element of the carrier of L. The functor $v \cdot p$ yields a sequence of L and is defined by:

(Def. 3) For every natural number n holds $(v \cdot p)(n) = v \cdot p(n)$.

Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, let p be a Polynomial of L, and let v be an element of the carrier of L. Observe that $v \cdot p$ is finite-Support.

We now state several propositions:

(25) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p be a Polynomial of L. Then $\mathrm{len}(0_L \cdot p) = 0$.

(26) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non empty double loop structure, p be a Polynomial of L, and v be an element of the carrier of L. If $v \neq 0_L$, then $\mathrm{len}(v \cdot p) = \mathrm{len}\, p$.

(27) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and p be a sequence of L. Then $0_L \cdot p = \mathbf{0}.\, L$.

(28) For every left unital non empty multiplicative loop structure L and for every sequence p of L holds $\mathbf{1}_L \cdot p = p$.

(29) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure and v be an element of the carrier of L. Then $v \cdot \mathbf{0}.\, L = \mathbf{0}.\, L$.

(30) Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure and v be an element of the carrier of L. Then $v \cdot \mathbf{1}.\, L = \langle v \rangle$.

(31) Let L be an add-associative right zeroed right complementable left unital distributive commutative associative field-like non empty double loop structure, p be a Polynomial of L, and v, x be elements of the carrier of L. Then $\mathrm{eval}(v \cdot p, x) = v \cdot \mathrm{eval}(p, x)$.

(32) Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and p be a Polynomial of L. Then $\mathrm{eval}(p, 0_L) = p(0)$.

Let L be a non empty zero structure and let z_0, z_1 be elements of the carrier of L. The functor $\langle z_0, z_1 \rangle$ yields a sequence of L and is defined by:

(Def. 4) $\langle z_0, z_1 \rangle = \mathbf{0}.\, L +\cdot (0, z_0) +\cdot (1, z_1)$.

The following propositions are true:

(33) Let L be a non empty zero structure and z_0 be an element of the carrier of L. Then $\langle z_0 \rangle(0) = z_0$ and for every natural number n such that $n \geqslant 1$ holds $\langle z_0 \rangle(n) = 0_L$.

(34) For every non empty zero structure L and for every element z_0 of the carrier of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle z_0 \rangle = 1$.

(35) For every non empty zero structure L holds $\langle 0_L \rangle = \mathbf{0}.\, L$.

(36) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital field-like non empty double loop structure and x, y be elements of the carrier of L. Then $\langle x \rangle * \langle y \rangle = \langle x \cdot y \rangle$.

(37) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty do-

uble loop structure, x be an element of the carrier of L, and n be a natural number. Then $\langle x\rangle^n = \langle \mathrm{power}_L(x, n)\rangle$.

(38) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and z_0, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0\rangle, x) = z_0$.

(39) Let L be a non empty zero structure and z_0, z_1 be elements of the carrier of L. Then $\langle z_0, z_1\rangle(0) = z_0$ and $\langle z_0, z_1\rangle(1) = z_1$ and for every natural number n such that $n \geqslant 2$ holds $\langle z_0, z_1\rangle(n) = 0_L$.

Let L be a non empty zero structure and let z_0, z_1 be elements of the carrier of L. One can verify that $\langle z_0, z_1\rangle$ is finite-Support.

The following propositions are true:

(40) For every non empty zero structure L and for all elements z_0, z_1 of the carrier of L holds $\mathrm{len}\langle z_0, z_1\rangle \leqslant 2$.

(41) For every non empty zero structure L and for all elements z_0, z_1 of the carrier of L such that $z_1 \neq 0_L$ holds $\mathrm{len}\langle z_0, z_1\rangle = 2$.

(42) For every non empty zero structure L and for every element z_0 of the carrier of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle z_0, 0_L\rangle = 1$.

(43) For every non empty zero structure L holds $\langle 0_L, 0_L\rangle = \mathbf{0}.\,L$.

(44) For every non empty zero structure L and for every element z_0 of the carrier of L holds $\langle z_0, 0_L\rangle = \langle z_0\rangle$.

(45) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, z_1\rangle, x) = z_0 + z_1 \cdot x$.

(46) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, 0_L\rangle, x) = z_0$.

(47) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle 0_L, z_1\rangle, x) = z_1 \cdot x$.

(48) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle z_0, \mathbf{1}_L\rangle, x) = z_0 + x$.

(49) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of the carrier of L. Then $\mathrm{eval}(\langle 0_L, \mathbf{1}_L\rangle, x) = x$.

3. Substitution in Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and let p, q be Polynomials of L. The functor $p[q]$ yielding a Polynomial of L is defined by the condition (Def. 5).

(Def. 5) There exists a finite sequence F of elements of the carrier of Polynom-Ring L such that $p[q] = \sum F$ and $\operatorname{len} F = \operatorname{len} p$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = p(n -' 1) \cdot q^{n -' 1}$.

One can prove the following propositions:

(50) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $(\mathbf{0}. L)[p] = \mathbf{0}. L$.

(51) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a Polynomial of L. Then $p[\mathbf{0}. L] = \langle p(0) \rangle$.

(52) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty double loop structure, p be a Polynomial of L, and x be an element of the carrier of L. Then $\operatorname{len}(p[\langle x \rangle]) \leqslant 1$.

(53) For every field L and for all Polynomials p, q of L such that $\operatorname{len} p \neq 0$ and $\operatorname{len} q > 1$ holds $\operatorname{len}(p[q]) = (\operatorname{len} p \cdot \operatorname{len} q - \operatorname{len} p - \operatorname{len} q) + 2$.

(54) Let L be a field, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\operatorname{eval}(p[q], x) = \operatorname{eval}(p, \operatorname{eval}(q, x))$.

4. Fundamental Theorem of Algebra

Let L be a unital non empty double loop structure, let p be a Polynomial of L, and let x be an element of the carrier of L. We say that x is a root of p if and only if:

(Def. 6) $\operatorname{eval}(p, x) = 0_L$.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. We say that p has roots if and only if:

(Def. 7) There exists an element x of the carrier of L such that x is a root of p.

The following proposition is true

(55) For every unital non empty double loop structure L holds $\mathbf{0}. L$ has roots.

Let L be a unital non empty double loop structure. One can verify that $\mathbf{0}.\,L$ has roots.

The following proposition is true

(56) Let L be a unital non empty double loop structure and x be an element of the carrier of L. Then x is a root of $\mathbf{0}.\,L$.

Let L be a unital non empty double loop structure. One can verify that there exists a Polynomial of L which has roots.

Let L be a unital non empty double loop structure. We say that L is algebraic-closed if and only if:

(Def. 8) For every Polynomial p of L such that $\operatorname{len} p > 1$ holds p has roots.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. The functor $\operatorname{Roots} p$ yields a subset of L and is defined by:

(Def. 9) For every element x of the carrier of L holds $x \in \operatorname{Roots} p$ iff x is a root of p.

Let L be a commutative associative left unital distributive field-like non empty double loop structure and let p be a Polynomial of L. The functor $\operatorname{NormPolynomial} p$ yielding a sequence of L is defined as follows:

(Def. 10) For every natural number n holds $(\operatorname{NormPolynomial} p)(n) = \frac{p(n)}{p(\operatorname{len} p -' 1)}$.

Let L be an add-associative right zeroed right complementable commutative associative left unital distributive field-like non empty double loop structure and let p be a Polynomial of L. Note that $\operatorname{NormPolynomial} p$ is finite-Support.

The following propositions are true:

(57) Let L be a commutative associative left unital distributive field-like non empty double loop structure and p be a Polynomial of L. If $\operatorname{len} p \neq 0$, then $(\operatorname{NormPolynomial} p)(\operatorname{len} p -' 1) = \mathbf{1}_L$.

(58) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds $\operatorname{len} \operatorname{NormPolynomial} p = \operatorname{len} p$.

(59) Let L be a field and p be a Polynomial of L. Suppose $\operatorname{len} p \neq 0$. Let x be an element of the carrier of L. Then $\operatorname{eval}(\operatorname{NormPolynomial} p, x) = \frac{\operatorname{eval}(p,x)}{p(\operatorname{len} p -' 1)}$.

(60) Let L be a field and p be a Polynomial of L. Suppose $\operatorname{len} p \neq 0$. Let x be an element of the carrier of L. Then x is a root of p if and only if x is a root of $\operatorname{NormPolynomial} p$.

(61) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds p has roots iff $\operatorname{NormPolynomial} p$ has roots.

(62) For every field L and for every Polynomial p of L such that $\operatorname{len} p \neq 0$ holds $\operatorname{Roots} p = \operatorname{Roots} \operatorname{NormPolynomial} p$.

(63) $\mathrm{id}_{\mathbb{C}}$ is continuous on $\mathbb{C}$.

(64) For every element x of $\mathbb{C}$ holds $\mathbb{C} \longmapsto x$ is continuous on $\mathbb{C}$.

Let L be a unital non empty groupoid, let x be an element of the carrier of L, and let n be a natural number. The functor FPower(x, n) yields a map from L into L and is defined as follows:

(Def. 11) For every element y of the carrier of L holds $(\text{FPower}(x, n))(y) = x \cdot \text{power}_L(y, n)$.

The following propositions are true:

(65) For every unital non empty groupoid L holds $\text{FPower}(1_L, 1) = \text{id}_{\text{the carrier of } L}$.

(66) $\text{FPower}(\mathbf{1}_{\mathbb{C}_{\text{F}}}, 2) = \text{id}_{\mathbb{C}}\, \text{id}_{\mathbb{C}}$.

(67) For every unital non empty groupoid L and for every element x of the carrier of L holds $\text{FPower}(x, 0) = (\text{the carrier of } L) \longmapsto x$.

(68) For every element x of the carrier of $\mathbb{C}_{\text{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\text{FPower}(x, 1) = x_1\, \text{id}_{\mathbb{C}}$.

(69) For every element x of the carrier of $\mathbb{C}_{\text{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\text{FPower}(x, 2) = x_1\, (\text{id}_{\mathbb{C}}\, \text{id}_{\mathbb{C}})$.

(70) Let x be an element of the carrier of $\mathbb{C}_{\text{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \text{FPower}(x, n)$ and $\text{FPower}(x, n+1) = f\, \text{id}_{\mathbb{C}}$.

(71) Let x be an element of the carrier of $\mathbb{C}_{\text{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \text{FPower}(x, n)$ and f is continuous on $\mathbb{C}$.

Let L be a unital non empty double loop structure and let p be a Polynomial of L. The functor Polynomial-Function(L, p) yields a map from L into L and is defined as follows:

(Def. 12) For every element x of the carrier of L holds $(\text{Polynomial-Function}(L, p))(x) = \text{eval}(p, x)$.

The following propositions are true:

(72) For every Polynomial p of $\mathbb{C}_{\text{F}}$ there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \text{Polynomial-Function}(\mathbb{C}_{\text{F}}, p)$ and f is continuous on $\mathbb{C}$.

(73) Let p be a Polynomial of $\mathbb{C}_{\text{F}}$. Suppose $\text{len}\, p > 2$ and $|p(\text{len}\, p -' 1)| = 1$. Let F be a finite sequence of elements of $\mathbb{R}$. Suppose $\text{len}\, F = \text{len}\, p$ and for every natural number n such that $n \in \text{dom}\, F$ holds $F(n) = |p(n -' 1)|$. Let z be an element of the carrier of $\mathbb{C}_{\text{F}}$. If $|z| > \sum F$, then $|\text{eval}(p, z)| > |p(0)| + 1$.

(74) Let p be a Polynomial of $\mathbb{C}_{\text{F}}$. Suppose $\text{len}\, p > 2$. Then there exists an element z_0 of the carrier of $\mathbb{C}_{\text{F}}$ such that for every element z of the carrier of $\mathbb{C}_{\text{F}}$ holds $|\text{eval}(p, z)| \geqslant |\text{eval}(p, z_0)|$.

(75) For every Polynomial p of $\mathbb{C}_{\text{F}}$ such that $\text{len}\, p > 1$ holds p has roots.

Let us note that $\mathbb{C}_{\text{F}}$ is algebraic-closed.

Let us mention that there exists a left unital right unital non empty double loop structure which is algebraic-closed, add-associative, right zeroed, right complementable, Abelian, commutative, associative, distributive, field-like, and non degenerated.

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Received August 21, 2000