FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

Lower Tolerance. Preliminaries to Wroclaw Taxonomy[1]

Mariusz Giero
University of Białystok

Roman Matuszewski
University of Białystok

Summary. The paper introduces some preliminary notions concerning the Wroclaw taxonomy according to [16]. The classifications and tolerances are defined and considered w.r.t. sets and metric spaces. We prove theorems showing various classifications based on tolerances.

MML Identifier: `TAXONOM1`.

The articles [14], [15], [20], [4], [9], [5], [6], [8], [12], [1], [13], [17], [19], [2], [23], [25], [24], [3], [18], [22], [21], [10], [11], and [7] provide the terminology and notation for this paper.

1. Preliminaries

In this paper A, X are non empty sets, f is a partial function from $[: X, X :]$ to $\mathbb{R}$, and a is a real number.

Let us note that there exists a real number which is non negative.

We now state a number of propositions:

(1) For every finite sequence p and for every natural number k such that $k+1 \in \operatorname{dom} p$ and $k \notin \operatorname{dom} p$ holds $k = 0$.

(2) Let p be a finite sequence and i, j be natural numbers. Suppose $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and for every natural number k such that $k \in \operatorname{dom} p$ and $k+1 \in \operatorname{dom} p$ holds $p(k) = p(k+1)$. Then $p(i) = p(j)$.

[1]This work has been partially supported by the European Community TYPES grant IST-1999-29001 and CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(3) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{dom} R = X$.

(4) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{rng} R = X$.

(5) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is reflexive in X.

(6) Let X, x, y be sets and R be a binary relation on X. Suppose R is reflexive in X. If R reduces x to y and $x \in X$, then $\langle x, y \rangle \in R^*$.

(7) Let X be a set and R be a binary relation on X. If R is reflexive in X and symmetric in X, then R^* is symmetric in X.

(8) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is transitive in X.

(9) Let X be a non empty set and R be a binary relation on X. Suppose R is reflexive in X and symmetric in X. Then R^* is an equivalence relation of X.

(10) For all binary relations R_1, R_2 on X such that $R_1 \subseteq R_2$ holds ${R_1}^* \subseteq {R_2}^*$.

(11) SmallestPartition(A) is finer than $\{A\}$.

2. The Notion of Classification

Let A be a non empty set. A subset of PARTITIONS(A) is called a classification of A if:

(Def. 1) For all partitions X, Y of A such that $X \in$ it and $Y \in$ it holds X is finer than Y or Y is finer than X.

One can prove the following propositions:

(12) $\{\{A\}\}$ is a classification of A.

(13) {SmallestPartition(A)} is a classification of A.

(14) For every subset S of PARTITIONS(A) such that $S = \{\{A\}$, SmallestPartition(A)$\}$ holds S is a classification of A.

Let A be a non empty set. A subset of PARTITIONS(A) is called a strong classification of A if:

(Def. 2) It is a classification of A and $\{A\} \in$ it and SmallestPartition(A) $\in$ it.

Next we state the proposition

(15) For every subset S of PARTITIONS(A) such that $S = \{\{A\}$, SmallestPartition(A)$\}$ holds S is a strong classification of A.

3. The Tolerance on a Non Empty Set

Let X be a non empty set, let f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and let a be a real number. The functor $\mathrm{T}_{\mathrm{l}}(f, a)$ yields a binary relation on X and is defined as follows:

(Def. 3) For all elements x, y of X holds $\langle x, y \rangle \in \mathrm{T}_{\mathrm{l}}(f, a)$ iff $f(x, y) \leqslant a$.

The following four propositions are true:

(16) If f is Reflexive and $a \geqslant 0$, then $\mathrm{T}_{\mathrm{l}}(f, a)$ is reflexive in X.

(17) If f is symmetric, then $\mathrm{T}_{\mathrm{l}}(f, a)$ is symmetric in X.

(18) If $a \geqslant 0$ and f is Reflexive and symmetric, then $\mathrm{T}_{\mathrm{l}}(f, a)$ is a tolerance of X.

(19) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and a_1, a_2 be real numbers. If $a_1 \leqslant a_2$, then $\mathrm{T}_{\mathrm{l}}(f, a_1) \subseteq \mathrm{T}_{\mathrm{l}}(f, a_2)$.

Let X be a set and let f be a partial function from $[: X, X :]$ to $\mathbb{R}$. We say that f is non-negative if and only if:

(Def. 4) For all elements x, y of X holds $f(x, y) \geqslant 0$.

We now state three propositions:

(20) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and x, y be sets. Suppose f is non-negative, Reflexive, and discernible. If $\langle x, y \rangle \in \mathrm{T}_{\mathrm{l}}(f, 0)$, then $x = y$.

(21) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and x be an element of X. If f is Reflexive and discernible, then $\langle x, x \rangle \in \mathrm{T}_{\mathrm{l}}(f, 0)$.

(22) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and a be a real number. Suppose $\mathrm{T}_{\mathrm{l}}(f, a)$ is reflexive in X and f is symmetric. Then $(\mathrm{T}_{\mathrm{l}}(f, a))^*$ is an equivalence relation of X.

4. The Partitions Defined by Lower Tolerance

Next we state several propositions:

(23) Let X be a non empty set and f be a partial function from $[: X, X :]$ to $\mathbb{R}$. Suppose f is non-negative, Reflexive, and discernible. Then $(\mathrm{T}_{\mathrm{l}}(f, 0))^* = \mathrm{T}_{\mathrm{l}}(f, 0)$.

(24) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T}_{\mathrm{l}}(f, 0))^*$ and f is non-negative, Reflexive, and discernible. Then $R = \triangle_X$.

(25) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T_l}(f, 0))^*$ and f is non-negative, Reflexive, and discernible. Then $\mathrm{Classes}\, R = \mathrm{SmallestPartition}(X)$.

(26) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[: X, X :]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\, f$ and $A \geqslant \max z$, then for all elements x, y of X holds $f(x, y) \leqslant A$.

(27) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[: X, X :]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. Suppose $z = \mathrm{rng}\, f$ and $A \geqslant \max z$. Let R be an equivalence relation of X. If $R = (\mathrm{T_l}(f, A))^*$, then $\mathrm{Classes}\, R = \{X\}$.

(28) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[: X, X :]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\, f$ and $A \geqslant \max z$, then $(\mathrm{T_l}(f, A))^* = \mathrm{T_l}(f, A)$.

5. The Classification on a Non Empty Set

Let X be a non empty set and let f be a partial function from $[: X, X :]$ to $\mathbb{R}$. The functor $\mathrm{FamClass}\, f$ yielding a subset of $\mathrm{PARTITIONS}(X)$ is defined by the condition (Def. 5).

(Def. 5) Let x be a set. Then $x \in \mathrm{FamClass}\, f$ if and only if there exists a non negative real number a and there exists an equivalence relation R of X such that $R = (\mathrm{T_l}(f, a))^*$ and $\mathrm{Classes}\, R = x$.

We now state four propositions:

(29) Let X be a non empty set, f be a partial function from $[: X, X :]$ to $\mathbb{R}$, and a be a non negative real number. If $\mathrm{T_l}(f, a)$ is reflexive in X and f is symmetric, then $\mathrm{FamClass}\, f$ is a non empty set.

(30) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[: X, X :]$ into $\mathbb{R}$. If f is symmetric and non-negative, then $\{X\} \in \mathrm{FamClass}\, f$.

(31) For every non empty set X and for every partial function f from $[: X, X :]$ to $\mathbb{R}$ holds $\mathrm{FamClass}\, f$ is a classification of X.

(32) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[: X, X :]$ into $\mathbb{R}$. Suppose $\mathrm{SmallestPartition}(X) \in \mathrm{FamClass}\, f$ and f is symmetric and non-negative. Then $\mathrm{FamClass}\, f$ is a strong classification of X.

6. The Classification on a Metric Space

Let M be a metric structure, let a be a real number, and let x, y be elements of the carrier of M. We say that x, y are in tolerance w.r.t. a if and only if:

(Def. 6) $\rho(x,y) \leqslant a$.

Let M be a non empty metric structure and let a be a real number. The functor $\mathrm{T_m}(M,a)$ yielding a binary relation on M is defined by:

(Def. 7) For all elements x, y of the carrier of M holds $\langle x, y\rangle \in \mathrm{T_m}(M,a)$ iff x, y are in tolerance w.r.t. a.

Next we state two propositions:

(33) For every non empty metric structure M and for every real number a holds $\mathrm{T_m}(M,a) = \mathrm{T_l}$(the distance of M, a).

(34) Let M be a non empty Reflexive symmetric metric structure, a be a real number, and T be a relation between the carrier of M and the carrier of M. If $T = \mathrm{T_m}(M,a)$ and $a \geqslant 0$, then T is a tolerance of the carrier of M.

Let M be a Reflexive symmetric non empty metric structure. The functor MetricFamClass M yielding a subset of PARTITIONS(the carrier of M) is defined by the condition (Def. 8).

(Def. 8) Let x be a set. Then $x \in$ MetricFamClass M if and only if there exists a non negative real number a and there exists an equivalence relation R of M such that $R = (\mathrm{T_m}(M,a))^*$ and Classes $R = x$.

The following propositions are true:

(35) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M = FamClass the distance of M.

(36) Let M be a non empty metric space and R be an equivalence relation of M. If $R = (\mathrm{T_m}(M,0))^*$, then Classes R = SmallestPartition(the carrier of M).

(37) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $\mathrm{T_m}(M,a) = \nabla_{\text{the carrier of } M}$.

(38) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $\mathrm{T_m}(M,a) = (\mathrm{T_m}(M,a))^*$.

(39) For every Reflexive symmetric bounded non empty metric structure M such that $a \geqslant \emptyset(\Omega_M)$ holds $(\mathrm{T_m}(M,a))^* = \nabla_{\text{the carrier of } M}$.

(40) Let M be a Reflexive symmetric bounded non empty metric structure, R be an equivalence relation of M, and a be a non negative real number. If $a \geqslant \emptyset(\Omega_M)$ and $R = (\mathrm{T_m}(M,a))^*$, then Classes $R = \{$the carrier of $M\}$.

Let M be a Reflexive symmetric triangle non empty metric structure and let C be a non empty bounded subset of M. Observe that $\emptyset C$ is non negative.

We now state three propositions:

(41) For every bounded non empty metric space M holds {the carrier of M} $\in$ MetricFamClass M.

(42) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M is a classification of the carrier of M.

(43) For every bounded non empty metric space M holds MetricFamClass M is a strong classification of the carrier of M.

Acknowledgments

The authors thank Prof. Andrzej Trybulec for his introduction to this topic. We thank Dr. Artur Kornilowicz for his advice on this article. We also thank Robert Milewski and Adam Naumowicz for their helpful comments.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. Minimal signature for partial algebra. *Formalized Mathematics*, 5(**3**):405–414, 1996.
[3] Grzegorz Bancerek. Reduction relations. *Formalized Mathematics*, 5(**4**):469–478, 1996.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Alicia de la Cruz. Totally bounded metric spaces. *Formalized Mathematics*, 2(**4**):559–562, 1991.
[11] Krzysztof Hryniewiecki. Relations of tolerance. *Formalized Mathematics*, 2(**1**):105–109, 1991.
[12] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[13] Shunichi Kobayashi and Kui Jia. A theory of partitions. Part I. *Formalized Mathematics*, 7(**2**):243–247, 1998.
[14] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Formalized Mathematics*, 4(**1**):109–113, 1993.
[15] Richard Krueger, Piotr Rudnicki, and Paul Shelley. Asymptotic notation. Part I: Theory. *Formalized Mathematics*, 9(**1**):135–142, 2001.
[16] Roman Matuszewski and Andrzej Trybulec. *Certain algorithm of classification in metric spaces*, volume V, Number 20 of *Mathematical Papers*. Warsaw University, Bialystok Campus, 1977.
[17] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[20] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[25] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

Received December 5, 2000