FORMALIZED MATHEMATICS
Volume 9, Number 4, 2001
University of Białystok

Zero-Based Finite Sequences

Tetsuya Tsunetou
Kyushu University

Grzegorz Bancerek
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

MML Identifier: AFINSQ_1.

The terminology and notation used in this paper are introduced in the following papers: [11], [4], [7], [6], [5], [1], [3], [2], [8], [12], [13], [10], and [9].

We follow the rules: k, n are natural numbers, x, y, z, y_1, y_2, X are sets, and f is a function.

One can prove the following propositions:

(1) $n \in n+1$.

(2) If $k \leqslant n$, then $k = k \cap n$.

(3) If $k = k \cap n$, then $k \leqslant n$.

(4) $n \cup \{n\} = n+1$.

(5) $\mathrm{Seg}\, n \subseteq n+1$.

(6) $n+1 = \{0\} \cup \mathrm{Seg}\, n$.

(7) For every function r holds r is finite and transfinite sequence-like iff there exists n such that $\mathrm{dom}\, r = n$.

Let us mention that there exists a function which is finite and transfinite sequence-like.

A finite 0-sequence is a finite transfinite sequence.

In the sequel p, q, r denote finite 0-sequences.

Observe that every set which is natural is also finite. Let us consider p. One can verify that $\mathrm{dom}\, p$ is natural.

Let us consider p. Then $\overline{\overline{p}}$ is a natural number and it can be characterized by the condition:

(Def. 1) $\overline{\overline{p}} = \mathrm{dom}\, p$.

We introduce $\mathrm{len}\, p$ as a synonym of $\overline{\overline{p}}$.

Let us consider p. Then $\mathrm{dom}\, p$ is a subset of $\mathbb{N}$.

Next we state the proposition

ISSN 1426–2630

(8) If there exists k such that $\mathrm{dom}\, f \subseteq k$, then there exists p such that $f \subseteq p$.

In this article we present several logical schemes. The scheme *XSeqEx* deals with a natural number $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists p such that $\mathrm{dom}\, p = \mathcal{A}$ and for every k such that $k \in \mathcal{A}$ holds $\mathcal{P}[k, p(k)]$

provided the following conditions are satisfied:

- For all k, y_1, y_2 such that $k \in \mathcal{A}$ and $\mathcal{P}[k, y_1]$ and $\mathcal{P}[k, y_2]$ holds $y_1 = y_2$, and
- For every k such that $k \in \mathcal{A}$ there exists x such that $\mathcal{P}[k, x]$.

The scheme *SeqLambda* deals with a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a finite 0-sequence p such that $\mathrm{len}\, p = \mathcal{A}$ and for every k such that $k \in \mathcal{A}$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

Next we state several propositions:

(9) If $z \in p$, then there exists k such that $k \in \mathrm{dom}\, p$ and $z = \langle k, p(k) \rangle$.

(10) If $\mathrm{dom}\, p = \mathrm{dom}\, q$ and for every k such that $k \in \mathrm{dom}\, p$ holds $p(k) = q(k)$, then $p = q$.

(11) If $\mathrm{len}\, p = \mathrm{len}\, q$ and for every k such that $k < \mathrm{len}\, p$ holds $p(k) = q(k)$, then $p = q$.

(12) $p \restriction n$ is a finite 0-sequence.

(13) If $\mathrm{rng}\, p \subseteq \mathrm{dom}\, f$, then $f \cdot p$ is a finite 0-sequence.

(14) If $k < \mathrm{len}\, p$ and $q = p \restriction k$, then $\mathrm{len}\, q = k$ and $\mathrm{dom}\, q = k$.

Let D be a set. Observe that there exists a transfinite sequence of elements of D which is finite.

Let D be a set. A finite 0-sequence of D is a finite transfinite sequence of elements of D.

We now state the proposition

(15) For every set D holds every finite 0-sequence of D is a partial function from $\mathbb{N}$ to D.

One can verify that $\emptyset$ is transfinite sequence-like.

Let D be a set. Observe that there exists a partial function from $\mathbb{N}$ to D which is finite and transfinite sequence-like.

In the sequel D is a set.

Next we state two propositions:

(16) For every finite 0-sequence p of D holds $p \restriction k$ is a finite 0-sequence of D.

(17) For every non empty set D there exists a finite 0-sequence p of D such that $\mathrm{len}\, p = k$.

One can verify that there exists a finite 0-sequence which is empty.

One can prove the following propositions:

(18) $\operatorname{len} p = 0$ iff $p = \emptyset$.

(19) For every set D holds $\emptyset$ is a finite 0-sequence of D.

Let D be a set. One can verify that there exists a finite 0-sequence of D which is empty.

Let us consider x. The functor $\langle_0 x\rangle$ yielding a set is defined as follows:

(Def. 2) $\langle_0 x\rangle = \{\langle 0, x\rangle\}$.

Let D be a set. The functor $\langle\rangle_D$ yields an empty finite 0-sequence of D and is defined by:

(Def. 3) $\langle\rangle_D = \emptyset$.

Let us consider p, q. Observe that $p \frown q$ is finite. Then $p \frown q$ can be characterized by the condition:

(Def. 4) $\operatorname{dom}(p \frown q) = \operatorname{len} p + \operatorname{len} q$ and for every k such that $k \in \operatorname{dom} p$ holds $(p \frown q)(k) = p(k)$ and for every k such that $k \in \operatorname{dom} q$ holds $(p \frown q)(\operatorname{len} p + k) = q(k)$.

The following propositions are true:

(20) $\operatorname{len}(p \frown q) = \operatorname{len} p + \operatorname{len} q$.

(21) If $\operatorname{len} p \leqslant k$ and $k < \operatorname{len} p + \operatorname{len} q$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(22) If $\operatorname{len} p \leqslant k$ and $k < \operatorname{len}(p \frown q)$, then $(p \frown q)(k) = q(k - \operatorname{len} p)$.

(23) If $k \in \operatorname{dom}(p \frown q)$, then $k \in \operatorname{dom} p$ or there exists n such that $n \in \operatorname{dom} q$ and $k = \operatorname{len} p + n$.

(24) For all transfinite sequences p, q holds $\operatorname{dom} p \subseteq \operatorname{dom}(p \frown q)$.

(25) If $x \in \operatorname{dom} q$, then there exists k such that $k = x$ and $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(26) If $k \in \operatorname{dom} q$, then $\operatorname{len} p + k \in \operatorname{dom}(p \frown q)$.

(27) $\operatorname{rng} p \subseteq \operatorname{rng}(p \frown q)$.

(28) $\operatorname{rng} q \subseteq \operatorname{rng}(p \frown q)$.

(29) $\operatorname{rng}(p \frown q) = \operatorname{rng} p \cup \operatorname{rng} q$.

(30) $(p \frown q) \frown r = p \frown (q \frown r)$.

(31) If $p \frown r = q \frown r$ or $r \frown p = r \frown q$, then $p = q$.

(32) $p \frown \emptyset = p$ and $\emptyset \frown p = p$.

(33) If $p \frown q = \emptyset$, then $p = \emptyset$ and $q = \emptyset$.

Let D be a set and let p, q be finite 0-sequences of D. Then $p \frown q$ is a transfinite sequence of elements of D.

Let us consider x. Then $\langle_0 x\rangle$ is a function and it can be characterized by the condition:

(Def. 5) $\operatorname{dom}\langle_0 x\rangle = 1$ and $\langle_0 x\rangle(0) = x$.

Let us consider x. One can verify that $\langle_0 x\rangle$ is function-like and relation-like.

Let us consider x. One can check that $\langle_0 x\rangle$ is finite and transfinite sequence-like.

One can prove the following proposition

(34) Suppose $p \frown q$ is a finite 0-sequence of D. Then p is a finite 0-sequence of D and q is a finite 0-sequence of D.

Let us consider x, y. The functor $\langle_0 x, y\rangle$ yielding a set is defined by:

(Def. 6) $\langle_0 x, y\rangle = \langle_0 x\rangle \frown \langle_0 y\rangle$.

Let us consider z. The functor $\langle_0 x, y, z\rangle$ yields a set and is defined by:

(Def. 7) $\langle_0 x, y, z\rangle = \langle_0 x\rangle \frown \langle_0 y\rangle \frown \langle_0 z\rangle$.

Let us consider x, y. One can check that $\langle_0 x, y\rangle$ is function-like and relation-like. Let us consider z. One can verify that $\langle_0 x, y, z\rangle$ is function-like and relation-like.

Let us consider x, y. One can check that $\langle_0 x, y\rangle$ is finite and transfinite sequence-like. Let us consider z. Observe that $\langle_0 x, y, z\rangle$ is finite and transfinite sequence-like.

One can prove the following propositions:

(35) $\langle_0 x\rangle = \{\langle 0, x\rangle\}$.

(36) $p = \langle_0 x\rangle$ iff $\operatorname{dom} p = 1$ and $\operatorname{rng} p = \{x\}$.

(37) $p = \langle_0 x\rangle$ iff $\operatorname{len} p = 1$ and $\operatorname{rng} p = \{x\}$.

(38) $p = \langle_0 x\rangle$ iff $\operatorname{len} p = 1$ and $p(0) = x$.

(39) $(\langle_0 x\rangle \frown p)(0) = x$.

(40) $(p \frown \langle_0 x\rangle)(\operatorname{len} p) = x$.

(41) $\langle_0 x, y, z\rangle = \langle_0 x\rangle \frown \langle_0 y, z\rangle$ and $\langle_0 x, y, z\rangle = \langle_0 x, y\rangle \frown \langle_0 z\rangle$.

(42) $p = \langle_0 x, y\rangle$ iff $\operatorname{len} p = 2$ and $p(0) = x$ and $p(1) = y$.

(43) $p = \langle_0 x, y, z\rangle$ iff $\operatorname{len} p = 3$ and $p(0) = x$ and $p(1) = y$ and $p(2) = z$.

(44) If $p \neq \emptyset$, then there exist q, x such that $p = q \frown \langle_0 x\rangle$.

Let D be a non empty set and let x be an element of D. Then $\langle_0 x\rangle$ is a finite 0-sequence of D.

The scheme *IndXSeq* concerns a unary predicate $\mathcal{P}$, and states that:

For every p holds $\mathcal{P}[p]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$, and
- For all p, x such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle_0 x\rangle]$.

We now state the proposition

(45) For all finite 0-sequences p, q, r, s such that $p \frown q = r \frown s$ and $\operatorname{len} p \leqslant \operatorname{len} r$ there exists a finite 0-sequence t such that $p \frown t = r$.

Let D be a set. The functor D^ω yields a set and is defined as follows:

(Def. 8) $x \in D^\omega$ iff x is a finite 0-sequence of D.

Let D be a set. One can check that D^ω is non empty.

One can prove the following propositions:

(46) $x \in D^\omega$ iff x is a finite 0-sequence of D.

(47) $\emptyset \in D^\omega$.

The scheme *SepSeq* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists X such that for every x holds $x \in X$ iff there exists p such that $p \in \mathcal{A}^\omega$ and $\mathcal{P}[p]$ and $x = p$

for all values of the parameters.

Let p be a finite 0-sequence and let i, x be sets. Note that $p +\cdot (i, x)$ is finite and transfinite sequence-like. We introduce $\mathrm{Replace}(p, i, x)$ as a synonym of $p +\cdot (i, x)$.

One can prove the following proposition

(48) Let p be a finite 0-sequence, i be a natural number, and x be a set. Then $\mathrm{len}\,\mathrm{Replace}(p, i, x) = \mathrm{len}\,p$ and if $i < \mathrm{len}\,p$, then $(\mathrm{Replace}(p, i, x))(i) = x$ and for every natural number j such that $j \neq i$ holds $(\mathrm{Replace}(p, i, x))(j) = p(j)$.

Let D be a non empty set, let p be a finite 0-sequence of D, let i be a natural number, and let a be an element of D. Then $\mathrm{Replace}(p, i, a)$ is a finite 0-sequence of D.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. Increasing and continuous ordinal sequences. *Formalized Mathematics*, 1(**4**):711–714, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[11] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[12] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[13] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received September 28, 2001