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Upper and Lower Sequence on the Cage. Part II[1]

Robert Milewski
University of Białystok

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The terminology and notation used here are introduced in the following articles: [29], [9], [22], [10], [1], [3], [27], [5], [25], [4], [16], [20], [15], [17], [19], [12], [21], [6], [8], [14], [23], [7], [2], [13], [30], [18], [26], [28], [24], and [11].

In this paper n is a natural number.

Let us note that there exists a finite sequence which is trivial.

The following proposition is true

(1) For every trivial finite sequence f holds f is empty or there exists a set x such that $f = \langle x \rangle$.

Let p be a non trivial finite sequence. Observe that $\mathrm{Rev}(p)$ is non trivial.

We now state four propositions:

(2) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be a set. Suppose f is a sequence which elements belong to G. Then $f -: p$ is a sequence which elements belong to G.

(3) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be an element of D. Suppose $p \in \mathrm{rng}\, f$. Suppose f is a sequence which elements belong to G. Then $f :- p$ is a sequence which elements belong to G.

(4) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{UpperSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

(5) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{LowerSeq}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$.

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Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. Note that UpperSeq(C,n) is standard and LowerSeq(C,n) is standard.

One can prove the following propositions:

(6) Let G be a column **Y**-constant line **Y**-increasing matrix over $\mathcal{E}^2_{\rm T}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{2}} = (G \circ (i_2, j_2))_{\mathbf{2}}$, then $j_1 = j_2$.

(7) Let G be a line **X**-constant column **X**-increasing matrix over $\mathcal{E}^2_{\rm T}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1\rangle \in$ the indices of G and $\langle i_2, j_2\rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{1}} = (G \circ (i_2, j_2))_{\mathbf{1}}$, then $i_1 = i_2$.

(8) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds N-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(9) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds N-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(10) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds E-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(11) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds E-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(12) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds S-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(13) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds S-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(14) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds W-min $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(15) For every non trivial finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds W-max $\widetilde{\mathcal{L}}(f) \in \operatorname{rng} f$.

(16) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq$ N-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ N-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-max $\widetilde{\mathcal{L}}(f)$, then (N-min $\widetilde{\mathcal{L}}(f))_{\mathbf{1}} <$ (N-max $\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(17) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq$ N-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ N-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ N-max $\widetilde{\mathcal{L}}(f)$, then N-min $\widetilde{\mathcal{L}}(f) \neq$ N-max $\widetilde{\mathcal{L}}(f)$.

(18) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq$ S-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ S-max $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-max $\widetilde{\mathcal{L}}(f)$, then (S-min $\widetilde{\mathcal{L}}(f))_{\mathbf{1}} <$ (S-max $\widetilde{\mathcal{L}}(f))_{\mathbf{1}}$.

(19) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq$ S-min $\widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \neq$ S-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$

S-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ S-max $\widetilde{\mathcal{L}}(f)$, then S-min $\widetilde{\mathcal{L}}(f) \neq$ S-max $\widetilde{\mathcal{L}}(f)$.

(20) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ W-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ W-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ W-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ W-max $\widetilde{\mathcal{L}}(f)$, then $(\text{W-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{W-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(21) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ W-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ W-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ W-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ W-max $\widetilde{\mathcal{L}}(f)$, then W-min $\widetilde{\mathcal{L}}(f) \neq$ W-max $\widetilde{\mathcal{L}}(f)$.

(22) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ E-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ E-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ E-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ E-max $\widetilde{\mathcal{L}}(f)$, then $(\text{E-min}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}} < (\text{E-max}\,\widetilde{\mathcal{L}}(f))_{\mathbf{2}}$.

(23) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f_1 \neq$ E-min $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ E-min $\widetilde{\mathcal{L}}(f)$ or $f_1 \neq$ E-max $\widetilde{\mathcal{L}}(f)$ and $f_{\mathrm{len}\,f} \neq$ E-max $\widetilde{\mathcal{L}}(f)$, then E-min $\widetilde{\mathcal{L}}(f) \neq$ E-max $\widetilde{\mathcal{L}}(f)$.

(24) Let D be a non empty set, f be a finite sequence of elements of D, and p, q be elements of D. If $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $q \leftarrow f \leqslant p \leftarrow f$, then $(f -: p) :- q = (f :- q) -: p$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) -:$ W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n) :-$ W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \{$N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(26) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LowerSeq}(C,n) = ((\mathrm{Cage}(C,n))^{\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) -:$ W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(27) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds (W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n) = 1$.

(28) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds (W-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n) <$ (W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n)$.

(29) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds (W-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n) \leqslant$ (N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n)$.

(30) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds (N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n) <$ (N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n)$.

(31) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds (N-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n) \leqslant$ (E-max $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \leftarrow \mathrm{UpperSeq}(C,n)$.

(32) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{UpperSeq}(C,n) = \text{len}\,\text{UpperSeq}(C,n)$.

(33) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) = 1$.

(34) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{E-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) < (\text{E-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n)$.

(35) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{E-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) \leqslant (\text{S-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n)$.

(36) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{S-max}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) < (\text{S-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n)$.

(37) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{S-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) \leqslant (\text{W-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n)$.

(38) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(\text{W-min}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))) \hookleftarrow \text{LowerSeq}(C,n) = \text{len}\,\text{LowerSeq}(C,n)$.

(39) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $((\text{UpperSeq}(C,n))_2)_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(40) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $((\text{LowerSeq}(C,n))_2)_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n))$.

(41) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\text{W-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n)) + \text{E-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \text{W-bound}\,C + \text{E-bound}\,C$.

(42) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\text{S-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n)) + \text{N-bound}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n)) = \text{S-bound}\,C + \text{N-bound}\,C$.

(43) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n, i be natural numbers. If $1 \leqslant i$ and $i \leqslant \text{width}\,\text{Gauge}(C,n)$ and $n > 0$, then $(\text{Gauge}(C,n) \circ (\text{Center}\,\text{Gauge}(C,n), i))_{\mathbf{1}} = \frac{\text{W-bound}\,C + \text{E-bound}\,C}{2}$.

(44) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n, i be natural numbers. If $1 \leqslant i$ and $i \leqslant \text{len}\,\text{Gauge}(C,n)$ and $n > 0$, then $(\text{Gauge}(C,n) \circ (i, \text{Center}\,\text{Gauge}(C,n)))_{\mathbf{2}} = \frac{\text{S-bound}\,C + \text{N-bound}\,C}{2}$.

(45) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \text{len}\,f$ and $1 \leqslant k_2$ and $k_2 \leqslant \text{len}\,f$ and $f_1 \in \widetilde{\mathcal{L}}(\text{mid}(f, k_1, k_2))$, then $k_1 = 1$ or $k_2 = 1$.

(46) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leqslant k_1$ and $k_1 \leqslant \text{len}\,f$ and $1 \leqslant k_2$ and $k_2 \leqslant \text{len}\,f$ and $f_{\text{len}\,f} \in \widetilde{\mathcal{L}}(\text{mid}(f, k_1, k_2))$, then $k_1 = \text{len}\,f$ or $k_2 = \text{len}\,f$.

(47) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Then $\mathrm{rng}\,\mathrm{UpperSeq}(C,n) \subseteq \mathrm{rng}\,\mathrm{Cage}(C,n)$ and $\mathrm{rng}\,\mathrm{LowerSeq}(C,n) \subseteq \mathrm{rng}\,\mathrm{Cage}(C,n)$.

(48) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{UpperSeq}(C,n)$ is a h.c. for $\mathrm{Cage}(C,n)$.

(49) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{Rev}(\mathrm{LowerSeq}(C,n))$ is a h.c. for $\mathrm{Cage}(C,n)$.

(50) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 < i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathrm{Gauge}(C,n) \circ (i,1) \notin \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$.

(51) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 \leqslant i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathrm{Gauge}(C,n) \circ (i, \mathrm{width}\,\mathrm{Gauge}(C,n)) \notin \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$.

(52) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 < i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathrm{Gauge}(C,n) \circ (i,1) \notin \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(53) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 \leqslant i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$, then $\mathrm{Gauge}(C,n) \circ (i, \mathrm{width}\,\mathrm{Gauge}(C,n)) \notin \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(54) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n)\circ(i,j) \in \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n)\circ(i,1), \mathrm{Gauge}(C,n)\circ(i,j))$ meets $\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(55) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)),$ $\mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) + \text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}) \in \mathrm{rng}\,\mathrm{UpperSeq}(C,n)$.

(56) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)),$ $\mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) + \text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}) \in \mathrm{rng}\,\mathrm{LowerSeq}(C,n)$.

(57) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \mathrm{rng}\,f$ holds $\downharpoonleft f, p = \mathrm{mid}(f, 1, p \looparrowleft f)$.

(58) Let f be a S-sequence in $\mathbb{R}^2$ and Q be a closed subset of $\mathcal{E}^2_{\rm T}$. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and $f_1 \notin Q$ and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\,f}, Q) \in \mathrm{rng}\,f$. Then $\widetilde{\mathcal{L}}(\mathrm{mid}(f, 1, (\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\,f}, Q)) \looparrowleft f)) \cap Q = \{\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\,f}, Q)\}$.

(59) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$.

Let k be a natural number. Suppose $1 \leqslant k$ and $k < (\mathrm{FPoint}(\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2})) \hookleftarrow \mathrm{UpperSeq}(C,n)$. Then $((\mathrm{UpperSeq}(C,n))_k)_{\mathbf{1}} < \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}$.

(60) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let k be a natural number. Suppose $1 \leqslant k$ and $k < (\mathrm{FPoint}(\widetilde{\mathcal{L}}(\mathrm{Rev}(\mathrm{LowerSeq}(C,n))), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2})) \hookleftarrow \mathrm{Rev}(\mathrm{LowerSeq}(C,n))$. Then $((\mathrm{Rev}(\mathrm{LowerSeq}(C,n)))_k)_{\mathbf{1}} < \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}$.

(61) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $q \in \mathrm{rng}\,\mathrm{mid}(\mathrm{UpperSeq}(C,n), 2, (\mathrm{FPoint}(\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2})) \hookleftarrow \mathrm{UpperSeq}(C,n))$. Then $q_{\mathbf{1}} \leqslant \frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}$.

(62) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Then $(\mathrm{FPoint}(\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}))_{\mathbf{2}} > (\mathrm{LPoint}(\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)), \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{VerticalLine}\,\frac{\text{W-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))+\text{E-bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}{2}))_{\mathbf{2}}$.

(63) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(64) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(65) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,1), \mathrm{Gauge}(C,n) \circ (i,j))$ meets $\mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

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