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Yet Another Construction of Free Algebra

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The notation and terminology used here are introduced in the following papers: [27], [21], [10], [15], [14], [9], [12], [8], [13], [23], [20], [6], [25], [11], [16], [7], [24], [17], [18], [19], [28], [29], [26], [22], [1], [3], [4], [5], and [2].

In this paper X, x, z are sets.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S. Observe that $\bigcup$(the sorts of A) is non empty.

Let S be a non empty non void many sorted signature and let A be a non empty algebra over S.

(Def. 1) An element of $\bigcup$(the sorts of A) is said to be an element of A.

We now state two propositions:

(1) For every function f such that $X \subseteq \operatorname{dom} f$ and f is one-to-one holds $f^{-1}(f^{\circ}X) = X$.

(2) Let I be a set, A be a many sorted set indexed by I, and F be a many sorted function indexed by I. If F is "1-1" and $A \subseteq \operatorname{dom}_{\kappa} F(\kappa)$, then $F^{-1}(F \circ A) = A$.

Let S be a non void signature and let X be a many sorted set indexed by the carrier of S. The functor $\mathrm{Free}_S(X)$ yields a strict algebra over S and is defined by:

(Def. 2) There exists a subset A of Free($X \cup$ ((the carrier of S) $\longmapsto \{0\}$)) such that $\mathrm{Free}_S(X) = \mathrm{Gen}(A)$ and $A =$ (Reverse($X \cup$ ((the carrier of S) $\longmapsto \{0\}$)))$^{-1}(X)$.

We now state four propositions:

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(3) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then $\langle x, s\rangle \in$ the carrier of DTConMSA(X) if and only if $x \in X(s)$.

(4) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then $x \in X(s)$ and $x \in Y(s)$ if and only if the root tree of $\langle x, s\rangle \in ((\text{Reverse}(Y))^{-1}(X))(s)$.

(5) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. If $x \in X(s)$, then the root tree of $\langle x, s\rangle \in$ (the sorts of $\text{Free}_S(X))(s)$.

(6) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, and o be an operation symbol of S. Suppose $\text{Arity}(o) = \emptyset$. Then the root tree of $\langle o$, the carrier of $S\rangle \in$ (the sorts of $\text{Free}_S(X)$)(the result sort of o).

Let S be a non void signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Observe that $\text{Free}_S(X)$ is non empty.

One can prove the following three propositions:

(7) Let S be a non void signature and X be a non-empty many sorted set indexed by the carrier of S. Then x is an element of Free(X) if and only if x is a term of S over X.

(8) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, s be a sort symbol of S, and x be a term of S over X. Then $x \in$ (the sorts of Free(X))(s) if and only if the sort of $x = s$.

(9) Let S be a non void signature and X be a non empty yielding many sorted set indexed by the carrier of S. Then every element of $\text{Free}_S(X)$ is a term of S over $X \cup$ ((the carrier of S) $\longmapsto \{0\}$).

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Note that every element of $\text{Free}_S(X)$ is relation-like and function-like.

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Note that every element of $\text{Free}_S(X)$ is finite and decorated tree-like.

Let S be a non empty non void many sorted signature and let X be a non empty yielding many sorted set indexed by the carrier of S. Observe that every element of $\text{Free}_S(X)$ is finite-branching.

One can check that every decorated tree is non empty.

Let S be a many sorted signature and let t be a non empty binary relation. The functor $\text{Var}_S\, t$ yields a many sorted set indexed by the carrier of S and is defined as follows:

(Def. 3) For every set s such that $s \in$ the carrier of S holds $(\text{Var}_S\, t)(s) = \{a_{\mathbf{1}}; a$

ranges over elements of $\operatorname{rng} t : a_{\mathbf{2}} = s\}$.

Let S be a many sorted signature, let X be a many sorted set indexed by the carrier of S, and let t be a non empty binary relation. The functor $\mathrm{Var}_X\, t$ yielding a many sorted subset indexed by X is defined by:

(Def. 4) $\mathrm{Var}_X\, t = X \cap \mathrm{Var}_S\, t$.

We now state several propositions:

(10) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, t be a non empty binary relation, and V be a many sorted subset indexed by X. Then $V = \mathrm{Var}_X\, t$ if and only if for every set s such that $s \in$ the carrier of S holds $V(s) = X(s) \cap \{a_{\mathbf{1}}; a$ ranges over elements of $\operatorname{rng} t : a_{\mathbf{2}} = s\}$.

(11) Let S be a many sorted signature and s, x be sets. Then

(i) if $s \in$ the carrier of S, then $(\mathrm{Var}_S\,(\text{the root tree of } \langle x, s\rangle))(s) = \{x\}$, and

(ii) for every set s' such that $s' \neq s$ or $s \notin$ the carrier of S holds $(\mathrm{Var}_S\,(\text{the root tree of } \langle x, s\rangle))(s') = \emptyset$.

(12) Let S be a many sorted signature and s be a set. Suppose $s \in$ the carrier of S. Let p be a decorated tree yielding finite sequence. Then $x \in (\mathrm{Var}_S(\langle z, \text{the carrier of } S\rangle\text{-tree}(p)))(s)$ if and only if there exists a decorated tree t such that $t \in \operatorname{rng} p$ and $x \in (\mathrm{Var}_S\, t)(s)$.

(13) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, and s, x be sets. Then

(i) if $x \in X(s)$, then $(\mathrm{Var}_X\,(\text{the root tree of } \langle x, s\rangle))(s) = \{x\}$, and

(ii) for every set s' such that $s' \neq s$ or $x \notin X(s)$ holds $(\mathrm{Var}_X\,(\text{the root tree of } \langle x, s\rangle))(s') = \emptyset$.

(14) Let S be a many sorted signature, X be a many sorted set indexed by the carrier of S, and s be a set. Suppose $s \in$ the carrier of S. Let p be a decorated tree yielding finite sequence. Then $x \in (\mathrm{Var}_X(\langle z, \text{the carrier of } S\rangle\text{-tree}(p)))(s)$ if and only if there exists a decorated tree t such that $t \in \operatorname{rng} p$ and $x \in (\mathrm{Var}_X\, t)(s)$.

(15) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_S\, t \subseteq X$.

Let S be a non void signature, let X be a non-empty many sorted set indexed by the carrier of S, and let t be a term of S over X. The functor Var_t yielding a many sorted subset indexed by X is defined by:

(Def. 5) $\mathrm{Var}_t = \mathrm{Var}_S\, t$.

The following proposition is true

(16) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_t = \mathrm{Var}_X\, t$.

Let S be a non void signature, let Y be a non-empty many sorted set indexed by the carrier of S, and let X be a many sorted set indexed by the carrier of S. The functor $S\text{-Terms}^Y(X)$ yielding a subset of $\mathrm{Free}(Y)$ is defined as follows:

(Def. 6) For every sort symbol s of S holds $(S\text{-Terms}^Y(X))(s) = \{t; t$ ranges over terms of S over Y: the sort of $t = s \ \wedge\ \mathrm{Var}_t \subseteq X\}$.

One can prove the following propositions:

(17) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. If $x \in (S\text{-Terms}^Y(X))(s)$, then x is a term of S over Y.

(18) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, t be a term of S over Y, and s be a sort symbol of S. If $t \in (S\text{-Terms}^Y(X))(s)$, then the sort of $t = s$ and $\mathrm{Var}_t \subseteq X$.

(19) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then the root tree of $\langle x,\ s\rangle \in (S\text{-Terms}^Y(X))(s)$ if and only if $x \in X(s)$ and $x \in Y(s)$.

(20) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, X be a many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be an argument sequence of $\mathrm{Sym}(o, Y)$. Then $\mathrm{Sym}(o, Y)\text{-tree}(p) \in (S\text{-Terms}^Y(X))$(the result sort of o) if and only if $\mathrm{rng}\, p \subseteq \bigcup(S\text{-Terms}^Y(X))$.

(21) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and A be a subset of $\mathrm{Free}(X)$. Then A is operations closed if and only if for every operation symbol o of S and for every argument sequence p of $\mathrm{Sym}(o, X)$ such that $\mathrm{rng}\, p \subseteq \bigcup A$ holds $\mathrm{Sym}(o, X)\text{-tree}(p) \in A$(the result sort of o).

(22) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, and X be a many sorted set indexed by the carrier of S. Then $S\text{-Terms}^Y(X)$ is operations closed.

(23) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, and X be a many sorted set indexed by the carrier of S. Then $(\mathrm{Reverse}(Y))^{-1}(X) \subseteq S\text{-Terms}^Y(X)$.

(24) Let S be a non void signature, X be a many sorted set indexed by the carrier of S, t be a term of S over $X \cup (($the carrier of $S) \longmapsto \{0\})$, and s be a sort symbol of S. If $t \in (S\text{-Terms}^{X \cup ((\text{the carrier of } S) \longmapsto \{0\})}(X))(s)$, then $t \in$ (the sorts of $\mathrm{Free}_S(X))(s)$.

(25) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then the sorts of $\mathrm{Free}_S(X)$ $=$

S-Terms$^{X \cup ((\text{the carrier of } S) \longmapsto \{0\})}(X)$.

(26) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then $\mathrm{Free}(X \cup ((\text{the carrier of } S) \longmapsto \{0\}))\restriction(S\text{-Terms}^{X \cup ((\text{the carrier of } S) \longmapsto \{0\})}(X)) = \mathrm{Free}_S(X)$.

(27) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, A be a subalgebra of $\mathrm{Free}(X)$, and B be a subalgebra of $\mathrm{Free}(Y)$. Suppose the sorts of $A =$ the sorts of B. Then the algebra of $A =$ the algebra of B.

(28) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, Y be a many sorted set indexed by the carrier of S, and t be an element of $\mathrm{Free}_S(X)$. Then $\mathrm{Var}_S\, t \subseteq X$.

(29) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a term of S over X. Then $\mathrm{Var}_t \subseteq X$.

(30) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, t_1 be a term of S over X, and t_2 be a term of S over Y. If $t_1 = t_2$, then the sort of $t_1 =$ the sort of t_2.

(31) Let S be a non void signature, X, Y be non-empty many sorted sets indexed by the carrier of S, and t be a term of S over Y. If $\mathrm{Var}_t \subseteq X$, then t is a term of S over X.

(32) Let S be a non void signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\mathrm{Free}_S(X) = \mathrm{Free}(X)$.

(33) Let S be a non void signature, Y be a non-empty many sorted set indexed by the carrier of S, t be a term of S over Y, and p be an element of $\mathrm{dom}\, t$. Then $\mathrm{Var}_{t\restriction p} \subseteq \mathrm{Var}_t$.

(34) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, t be an element of $\mathrm{Free}_S(X)$, and p be an element of $\mathrm{dom}\, t$. Then $t\restriction p$ is an element of $\mathrm{Free}_S(X)$.

(35) Let S be a non void signature, X be a non-empty many sorted set indexed by the carrier of S, t be a term of S over X, and a be an element of $\mathrm{rng}\, t$. Then $a = \langle a_{\mathbf{1}}, a_{\mathbf{2}} \rangle$.

(36) Let S be a non void signature, X be a non empty yielding many sorted set indexed by the carrier of S, t be an element of $\mathrm{Free}_S(X)$, and s be a sort symbol of S. Then

(i) if $x \in (\mathrm{Var}_S\, t)(s)$, then $\langle x, s \rangle \in \mathrm{rng}\, t$, and

(ii) if $\langle x, s \rangle \in \mathrm{rng}\, t$, then $x \in (\mathrm{Var}_S\, t)(s)$ and $x \in X(s)$.

(37) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Suppose that for every sort symbol s of S such that $X(s) = \emptyset$ there exists an operation symbol o of S such that the result sort of $o = s$ and $\mathrm{Arity}(o) = \emptyset$. Then $\mathrm{Free}_S(X)$ is non-empty.

(38) Let S be a non void signature, A be an algebra over S, B be a subalgebra

of A, and o be an operation symbol of S. Then $\mathrm{Args}(o, B) \subseteq \mathrm{Args}(o, A)$.

(39) For every non void signature S and for every feasible algebra A over S holds every subalgebra of A is feasible.

The following proposition is true

(40) Let S be a non void signature and X be a many sorted set indexed by the carrier of S. Then $\mathrm{Free}_S(X)$ is feasible and free.

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