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More on Multivariate Polynomials: Monomials and Constant Polynomials

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Summary. In this article we give some technical concepts for multivariate polynomials with arbitrary number of variables. Monomials and constant polynomials are introduced and their properties with respect to the eval functor are shown. In addition, the multiplication of polynomials with coefficients is defined and investigated.

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The notation and terminology used here are introduced in the following articles: [6], [10], [15], [13], [1], [12], [7], [9], [8], [2], [14], [3], [11], [4], and [5].

1. Preliminaries

Let us note that there exists a non empty zero structure which is non trivial.

Let us observe that every zero structure which is non trivial is also non empty.

Let us mention that there exists a non trivial double loop structure which is Abelian, left zeroed, right zeroed, add-associative, right complementable, unital, associative, commutative, distributive, and integral domain-like.

Let R be a non empty zero structure and let a be an element of R. We say that a is non-zero if and only if:

(Def. 1) $a \neq 0_R$.

Let R be a non trivial zero structure. Note that there exists an element of R which is non-zero.

Let X be a set, let R be a non empty zero structure, and let p be a series of X, R. We say that p is non-zero if and only if:

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(Def. 2) $p \neq 0\_(X, R)$.

Let X be a set and let R be a non trivial zero structure. One can check that there exists a series of X, R which is non-zero.

Let n be an ordinal number and let R be a non trivial zero structure. Note that there exists a polynomial of n, R which is non-zero.

The following two propositions are true:

(1) Let X be a set, R be a non empty zero structure, and s be a series of X, R. Then $s = 0\_(X, R)$ if and only if $\mathrm{Support}\, s = \emptyset$.

(2) Let X be a set and R be a non empty zero structure. Then R is non trivial if and only if there exists a series s of X, R such that $\mathrm{Support}\, s \neq \emptyset$.

Let X be a set and let b be a bag of X. We say that b is univariate if and only if:

(Def. 3) There exists an element u of X such that $\mathrm{support}\, b = \{u\}$.

Let X be a non empty set. Note that there exists a bag of X which is univariate.

Let X be a non empty set. Note that every bag of X which is univariate is also non empty.

2. Polynomials without Variables

We now state three propositions:

(3) For every bag b of $\emptyset$ holds $b = \mathrm{EmptyBag}\, \emptyset$.

(4) Let L be a right zeroed add-associative right complementable well unital distributive non trivial double loop structure, p be a polynomial of $\emptyset$, L, and x be a function from $\emptyset$ into L. Then $\mathrm{eval}(p, x) = p(\mathrm{EmptyBag}\, \emptyset)$.

(5) Let L be a right zeroed add-associative right complementable well unital distributive non trivial double loop structure. Then $\text{Polynom-Ring}(\emptyset, L)$ is ring isomorphic to L.

3. Monomials

Let X be a set, let L be a non empty zero structure, and let p be a series of X, L. We say that p is monomial-like if and only if:

(Def. 4) There exists a bag b of X such that for every bag b' of X such that $b' \neq b$ holds $p(b') = 0_L$.

Let X be a set and let L be a non empty zero structure. Note that there exists a series of X, L which is monomial-like.

Let X be a set and let L be a non empty zero structure. A monomial of X, L is a monomial-like series of X, L.

Let X be a set and let L be a non empty zero structure. One can check that every series of X, L which is monomial-like is also finite-Support.

The following proposition is true

(6) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a monomial of X, L if and only if $\mathrm{Support}\, p = \emptyset$ or there exists a bag b of X such that $\mathrm{Support}\, p = \{b\}$.

Let X be a set, let L be a non empty zero structure, let a be an element of L, and let b be a bag of X. The functor $\mathrm{Monom}(a, b)$ yields a monomial of X, L and is defined as follows:

(Def. 5) $\mathrm{Monom}(a, b) = 0\_(X, L) +\cdot (b, a)$.

Let X be a set, let L be a non empty zero structure, and let m be a monomial of X, L. The functor $\mathrm{term}\, m$ yielding a bag of X is defined by:

(Def. 6) $m(\mathrm{term}\, m) \neq 0_L$ or $\mathrm{Support}\, m = \emptyset$ and $\mathrm{term}\, m = \mathrm{EmptyBag}\, X$.

Let X be a set, let L be a non empty zero structure, and let m be a monomial of X, L. The functor $\mathrm{coefficient}\, m$ yields an element of L and is defined by:

(Def. 7) $\mathrm{coefficient}\, m = m(\mathrm{term}\, m)$.

One can prove the following propositions:

(7) For every set X and for every non empty zero structure L and for every monomial m of X, L holds $\mathrm{Support}\, m = \emptyset$ or $\mathrm{Support}\, m = \{\mathrm{term}\, m\}$.

(8) For every set X and for every non empty zero structure L and for every bag b of X holds $\mathrm{coefficient}\, \mathrm{Monom}(0_L, b) = 0_L$ and $\mathrm{term}\, \mathrm{Monom}(0_L, b) = \mathrm{EmptyBag}\, X$.

(9) Let X be a set, L be a non empty zero structure, a be an element of L, and b be a bag of X. Then $\mathrm{coefficient}\, \mathrm{Monom}(a, b) = a$.

(10) Let X be a set, L be a non trivial zero structure, a be a non-zero element of L, and b be a bag of X. Then $\mathrm{term}\, \mathrm{Monom}(a, b) = b$.

(11) For every set X and for every non empty zero structure L and for every monomial m of X, L holds $\mathrm{Monom}(\mathrm{coefficient}\, m, \mathrm{term}\, m) = m$.

(12) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, m be a monomial of n, L, and x be a function from n into L. Then $\mathrm{eval}(m, x) = \mathrm{coefficient}\, m \cdot \mathrm{eval}(\mathrm{term}\, m, x)$.

(13) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, a be an element of L, b be a bag of n, and x be a function from n into L. Then $\mathrm{eval}(\mathrm{Monom}(a, b), x) = a \cdot \mathrm{eval}(b, x)$.

4. Constant Polynomials

Let X be a set, let L be a non empty zero structure, and let p be a series of X, L. We say that p is constant if and only if:

(Def. 8) For every bag b of X such that $b \neq \mathrm{EmptyBag}\, X$ holds $p(b) = 0_L$.

Let X be a set and let L be a non empty zero structure. Observe that there exists a series of X, L which is constant.

Let X be a set and let L be a non empty zero structure. A constant polynomial of X, L is a constant series of X, L.

Let X be a set and let L be a non empty zero structure. One can check that every series of X, L which is constant is also monomial-like.

The following proposition is true

(14) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a constant polynomial of X, L if and only if $p = 0\_(X, L)$ or $\mathrm{Support}\, p = \{\mathrm{EmptyBag}\, X\}$.

Let X be a set and let L be a non empty zero structure. Observe that $0\_(X, L)$ is constant.

Let X be a set and let L be a unital non empty double loop structure. One can check that $1\_(X, L)$ is constant.

The following propositions are true:

(15) Let X be a set, L be a non empty zero structure, and c be a constant polynomial of X, L. Then $\mathrm{Support}\, c = \emptyset$ or $\mathrm{Support}\, c = \{\mathrm{EmptyBag}\, X\}$.

(16) Let X be a set, L be a non empty zero structure, and c be a constant polynomial of X, L. Then $\mathrm{term}\, c = \mathrm{EmptyBag}\, X$ and $\mathrm{coefficient}\, c = c(\mathrm{EmptyBag}\, X)$.

Let X be a set, let L be a non empty zero structure, and let a be an element of L. The functor $a\,\_(X, L)$ yielding a series of X, L is defined by:

(Def. 9) $a\,\_(X, L) = 0\_(X, L) +\cdot (\mathrm{EmptyBag}\, X, a)$.

Let X be a set, let L be a non empty zero structure, and let a be an element of L. Observe that $a\,\_(X, L)$ is constant.

We now state several propositions:

(17) Let X be a set, L be a non empty zero structure, and p be a series of X, L. Then p is a constant polynomial of X, L if and only if there exists an element a of L such that $p = a\,\_(X, L)$.

(18) Let X be a set, L be a non empty multiplicative loop with zero structure, and a be an element of L. Then $(a\,\_(X, L))(\mathrm{EmptyBag}\, X) = a$ and for every bag b of X such that $b \neq \mathrm{EmptyBag}\, X$ holds $(a\,\_(X, L))(b) = 0_L$.

(19) For every set X and for every non empty zero structure L holds $0_L\,\_(X, L) = 0\_(X, L)$.

(20) For every set X and for every unital non empty multiplicative loop with zero structure L holds $1_L\,\_(X, L) = 1\_(X, L)$.

(21) Let X be a set, L be a non empty zero structure, and a, b be elements of L. Then $a\,\_(X, L) = b\,\_(X, L)$ if and only if $a = b$.

(22) For every set X and for every non empty zero structure L and for every element a of L holds $\mathrm{Support}\,a\,\_(X, L) = \emptyset$ or $\mathrm{Support}\,a\,\_(X, L) = \{\mathrm{EmptyBag}\,X\}$.

(23) For every set X and for every non empty zero structure L and for every element a of L holds $\mathrm{term}\,a\,\_(X, L) = \mathrm{EmptyBag}\,X$ and $\mathrm{coefficient}\,a\,\_(X, L) = a$.

(24) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, c be a constant polynomial of n, L, and x be a function from n into L. Then $\mathrm{eval}(c, x) = \mathrm{coefficient}\,c$.

(25) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a\,\_(n, L), x) = a$.

5. Multiplication with Coefficients

Let X be a set, let L be a non empty multiplicative loop with zero structure, let p be a series of X, L, and let a be an element of L. The functor $a \cdot p$ yields a series of X, L and is defined by:

(Def. 10) For every bag b of X holds $(a \cdot p)(b) = a \cdot p(b)$.

The functor $p \cdot a$ yields a series of X, L and is defined by:

(Def. 11) For every bag b of X holds $(p \cdot a)(b) = p(b) \cdot a$.

Let X be a set, let L be a left zeroed right zeroed add-cancelable distributive non empty double loop structure, let p be a finite-Support series of X, L, and let a be an element of L. Note that $a \cdot p$ is finite-Support and $p \cdot a$ is finite-Support.

One can prove the following propositions:

(26) Let X be a set, L be a commutative non empty multiplicative loop with zero structure, p be a series of X, L, and a be an element of L. Then $a \cdot p = p \cdot a$.

(27) Let n be an ordinal number, L be an add-associative right complementable right zeroed left distributive non empty double loop structure, p be a series of n, L, and a be an element of L. Then $a \cdot p = (a\,\_(n, L)) * p$.

(28) Let n be an ordinal number, L be an add-associative right complementable right zeroed right distributive non empty double loop structure, p be a series of n, L, and a be an element of L. Then $p \cdot a = p * (a \_(n, L))$.

(29) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a \cdot p, x) = a \cdot \mathrm{eval}(p, x)$.

(30) Let n be an ordinal number, L be a left zeroed right zeroed add-left-cancelable add-associative right complementable unital associative integral domain-like distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(a \cdot p, x) = a \cdot \mathrm{eval}(p, x)$.

(31) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p \cdot a, x) = \mathrm{eval}(p, x) \cdot a$.

(32) Let n be an ordinal number, L be a left zeroed right zeroed add-left-cancelable add-associative right complementable unital associative commutative distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p \cdot a, x) = \mathrm{eval}(p, x) \cdot a$.

(33) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}((a \_(n, L)) * p, x) = a \cdot \mathrm{eval}(p, x)$.

(34) Let n be an ordinal number, L be an Abelian left zeroed right zeroed add-associative right complementable unital associative commutative distributive non trivial double loop structure, p be a polynomial of n, L, a be an element of L, and x be a function from n into L. Then $\mathrm{eval}(p * (a \_(n, L)), x) = \mathrm{eval}(p, x) \cdot a$.

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