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Robbins Algebras vs. Boolean Algebras[1]

Adam Grabowski
University of Białystok

Summary. In the early 1930s, Huntington proposed several axiom systems for Boolean algebras. Robbins slightly changed one of them and asked if the resulted system is still a basis for variety of Boolean algebras. The solution (afirmative answer) was given in 1996 by McCune with the help of automated theorem prover EQP/OTTER. Some simplified and restucturized versions of this proof are known. In our version of proof that all Robbins algebras are Boolean we use the results of McCune [5], Huntington [2, 4, 3] and Dahn [1].

MML Identifier: ROBBINS1.

The papers [7] and [6] provide the terminology and notation for this paper.

1. Preliminaries

We introduce complemented lattice structures which are extensions of ⊔-semi lattice structure and are systems

⟨ a carrier, a join operation, a complement operation ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, and the complement operation is a unary operation on the carrier.

We introduce ortholattice structures which are extensions of complemented lattice structure and lattice structure and are systems

⟨ a carrier, a join operation, a meet operation, a complement operation ⟩,

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, and the complement operation is a unary operation on the carrier.

The strict complemented lattice structure TrivComplLat is defined as follows:

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(Def. 1) $\mathrm{TrivComplLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

The strict ortholattice structure TrivOrtLat is defined by:

(Def. 2) $\mathrm{TrivOrtLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

Let us note that TrivComplLat is non empty and trivial and TrivOrtLat is non empty and trivial.

Let us mention that there exists an ortholattice structure which is strict, non empty, and trivial and there exists a complemented lattice structure which is strict, non empty, and trivial.

Let L be a non empty complemented lattice structure and let x be an element of the carrier of L. The functor x^{c} yielding an element of L is defined as follows:

(Def. 3) $x^{\mathrm{c}} = (\text{the complement operation of } L)(x)$.

Let L be a non empty complemented lattice structure and let x, y be elements of the carrier of L. We introduce $x + y$ as a synonym of $x \sqcup y$.

Let L be a non empty complemented lattice structure and let x, y be elements of the carrier of L. The functor $x * y$ yields an element of L and is defined by:

(Def. 4) $x * y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

Let L be a non empty complemented lattice structure. We say that L is Robbins if and only if:

(Def. 5) For all elements x, y of the carrier of L holds $((x + y)^{\mathrm{c}} + (x + y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = x$.

We say that L is Huntington if and only if:

(Def. 6) For all elements x, y of the carrier of L holds $(x^{\mathrm{c}} + y^{\mathrm{c}})^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}} = x$.

Let G be a non empty $\sqcup$-semi lattice structure. We say that G is join-idempotent if and only if:

(Def. 7) For every element x of the carrier of G holds $x \sqcup x = x$.

Let us observe that TrivComplLat is join-commutative join-associative Robbins Huntington and join-idempotent and TrivOrtLat is join-commutative join-associative Huntington and Robbins.

Let us mention that TrivOrtLat is meet-commutative meet-associative meet-absorbing and join-absorbing.

One can verify that there exists a non empty complemented lattice structure which is strict, join-associative, join-commutative, Robbins, join-idempotent, and Huntington.

Let us observe that there exists a non empty ortholattice structure which is strict, lattice-like, Robbins, and Huntington.

Let L be a join-commutative non empty complemented lattice structure and let x, y be elements of the carrier of L. Let us observe that the functor $x + y$ is commutative.

Next we state several propositions:

(1) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * b + a * b^{\rm c} = a$.

(2) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a be an element of the carrier of L. Then $a + a^{\rm c} = a^{\rm c} + (a^{\rm c})^{\rm c}$.

(3) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and x be an element of the carrier of L. Then $(x^{\rm c})^{\rm c} = x$.

(4) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + a^{\rm c} = b + b^{\rm c}$.

(5) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of the carrier of L such that for every element a of the carrier of L holds
$c + a = c$ and $a + a^{\rm c} = c$.

(6) Every join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure is upper-bounded.

One can verify that every non empty complemented lattice structure which is join-commutative, join-associative, join-idempotent, and Huntington is also upper-bounded.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $\top_L$ can be characterized by the condition:

(Def. 8) There exists an element a of the carrier of L such that $\top_L = a + a^{\rm c}$.

One can prove the following propositions:

(7) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of the carrier of L such that for every element a of the carrier of L holds
$c * a = c$ and $(a + a^{\rm c})^{\rm c} = c$.

(8) Let L be a join-commutative join-associative non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * b = b * a$.

Let L be a join-commutative join-associative non empty complemented lattice structure and let x, y be elements of the carrier of L. Let us note that the functor $x * y$ is commutative.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. The functor $\perp^{\rm C}_L$ yielding an element of L is defined as follows:

(Def. 9) For every element a of the carrier of L holds $\bot_L^{\rm C} * a = \bot_L^{\rm C}$.

One can prove the following propositions:

(9) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $\bot_L^{\rm C} = (a + a^{\rm c})^{\rm c}$.

(10) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $(\top_L)^{\rm c} = \bot_L^{\rm C}$ and $\top_L = (\bot_L^{\rm C})^{\rm c}$.

(11) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a^{\rm c} = b^{\rm c}$, then $a = b$.

(12) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + (b + b^{\rm c})^{\rm c} = a$.

(13) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a + a = a$.

Let us note that every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington is also join-idempotent.

One can prove the following propositions:

(14) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a + \bot_L^{\rm C} = a$.

(15) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * \top_L = a$.

(16) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * a^{\rm c} = \bot_L^{\rm C}$.

(17) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * (b * c) = (a * b) * c$.

(18) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + b = (a^{\rm c} * b^{\rm c})^{\rm c}$.

(19) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L. Then $a * a = a$.

(20) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of the carrier of L.

Then $a + \top_L = \top_L$.

(21) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a + a * b = a$.

(22) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. Then $a * (a + b) = a$.

(23) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a^{\rm c} + b = \top_L$ and $b^{\rm c} + a = \top_L$, then $a = b$.

(24) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of the carrier of L. If $a + b = \top_L$ and $a * b = \bot_L^{\rm C}$, then $a^{\rm c} = b$.

(25) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b * c + a * b * c^{\rm c} + a * b^{\rm c} * c + a * b^{\rm c} * c^{\rm c} + a^{\rm c} * b * c + a^{\rm c} * b * c^{\rm c} + a^{\rm c} * b^{\rm c} * c + a^{\rm c} * b^{\rm c} * c^{\rm c} = \top_L$.

(26) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then

(i) $a * c * (b * c^{\rm c}) = \bot_L^{\rm C}$,
(ii) $a * b * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$,
(iii) $a * b^{\rm c} * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$,
(iv) $a * b * c * (a^{\rm c} * b^{\rm c} * c) = \bot_L^{\rm C}$,
(v) $a * b * c^{\rm c} * (a^{\rm c} * b^{\rm c} * c^{\rm c}) = \bot_L^{\rm C}$, and
(vi) $a * b^{\rm c} * c * (a^{\rm c} * b * c) = \bot_L^{\rm C}$.

(27) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b + a * c = a * b * c + a * b * c^{\rm c} + a * b^{\rm c} * c$.

(28) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $(a * (b + c))^{\rm c} = a * b^{\rm c} * c^{\rm c} + a^{\rm c} * b * c + a^{\rm c} * b * c^{\rm c} + a^{\rm c} * b^{\rm c} * c + a^{\rm c} * b^{\rm c} * c^{\rm c}$.

(29) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a * b + a * c + (a * (b + c))^{\rm c} = \top_L$.

(30) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $(a * b + a * c) * (a * (b + c))^{\rm c} = \bot_L^{\rm C}$.

(31) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L.

Then $a * (b + c) = a * b + a * c$.

(32) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of the carrier of L. Then $a + b * c = (a + b) * (a + c)$.

2. Pre-Ortholattices

Let L be a non empty ortholattice structure. We say that L is well-complemented if and only if:

(Def. 10) For every element a of the carrier of L holds a^{c} is a complement of a.

Let us observe that TrivOrtLat is Boolean and well-complemented.

A pre-ortholattice is a lattice-like non empty ortholattice structure.

Let us mention that there exists a pre-ortholattice which is strict, Boolean, and well-complemented.

We now state two propositions:

(33) Let L be a distributive well-complemented pre-ortholattice and x be an element of the carrier of L. Then $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(34) Let L be a bounded distributive well-complemented pre-ortholattice and x, y be elements of the carrier of L. Then $x \sqcap y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

3. Correspondence between Boolean Pre-OrthoLattices and Boolean Lattices

Let L be a non empty complemented lattice structure. The functor $\mathrm{CLatt}\, L$ yielding a strict ortholattice structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of $\mathrm{CLatt}\, L$ = the carrier of L,

(ii) the join operation of $\mathrm{CLatt}\, L$ = the join operation of L,

(iii) the complement operation of $\mathrm{CLatt}\, L$ = the complement operation of L, and

(iv) for all elements a, b of the carrier of L holds (the meet operation of $\mathrm{CLatt}\, L)(a, b) = a * b$.

Let L be a non empty complemented lattice structure. One can verify that $\mathrm{CLatt}\, L$ is non empty.

Let L be a join-commutative non empty complemented lattice structure. One can check that $\mathrm{CLatt}\, L$ is join-commutative.

Let L be a join-associative non empty complemented lattice structure. One can check that $\mathrm{CLatt}\, L$ is join-associative.

Let L be a join-commutative join-associative non empty complemented lattice structure. Observe that CLatt L is meet-commutative.

The following proposition is true

(35) Let L be a non empty complemented lattice structure, a, b be elements of the carrier of L, and a', b' be elements of the carrier of CLatt L. If $a = a'$ and $b = b'$, then $a * b = a' \sqcap b'$ and $a + b = a' \sqcup b'$ and $a^{\rm c} = a'^{\rm c}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Observe that CLatt L is meet-associative join-absorbing and meet-absorbing.

Let L be a Huntington non empty complemented lattice structure. Note that CLatt L is Huntington.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Note that CLatt L is lower-bounded.

We now state the proposition

(36) For every join-commutative join-associative Huntington non empty complemented lattice structure L holds $\bot_L^{\rm C} = \bot_{{\rm CLatt}\, L}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. One can check that CLatt L is complemented distributive and bounded.

4. Proofs according to Bernd Ingo Dahn

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. We introduce $-x$ as a synonym of $x^{\rm c}$.

Let G be a join-commutative non empty complemented lattice structure. Let us observe that G is Huntington if and only if:

(Def. 12) For all elements x, y of the carrier of G holds $-(-x + -y) + -(x + -y) = y$.

Let G be a non empty complemented lattice structure. We say that G has idempotent element if and only if:

(Def. 13) There exists an element x of the carrier of G such that $x + x = x$.

In the sequel G is a Robbins join-associative join-commutative non empty complemented lattice structure and x, y, z are elements of the carrier of G.

Let G be a non empty complemented lattice structure and let x, y be elements of the carrier of G. The functor $\delta(x, y)$ yielding an element of G is defined by:

(Def. 14) $\delta(x, y) = -(-x + y)$.

Let G be a non empty complemented lattice structure and let x, y be elements of the carrier of G. The functor $\mathrm{Expand}(x,y)$ yields an element of G and is defined by:

(Def. 15) $\mathrm{Expand}(x,y) = \delta(x+y, \delta(x,y))$.

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. The functor x_0 yielding an element of G is defined by:

(Def. 16) $x_0 = -(-x+x)$.

The functor $2x$ yielding an element of G is defined as follows:

(Def. 17) $2x = x+x$.

Let G be a non empty complemented lattice structure and let x be an element of the carrier of G. The functor x_1 yielding an element of G is defined by:

(Def. 18) $x_1 = x_0 + x$.

The functor x_2 yields an element of G and is defined as follows:

(Def. 19) $x_2 = x_0 + 2x$.

The functor x_3 yields an element of G and is defined by:

(Def. 20) $x_3 = x_0 + (2x + x)$.

The functor x_4 yielding an element of G is defined as follows:

(Def. 21) $x_4 = x_0 + (2x + 2x)$.

We now state a number of propositions:

(37) $\delta(x+y, \delta(x,y)) = y$.

(38) $\mathrm{Expand}(x,y) = y$.

(39) $\delta(-x+y, z) = -(\delta(x,y)+z)$.

(40) $\delta(x,x) = x_0$.

(41) $\delta(2x, x_0) = x$.

(42) $\delta(x_2, x) = x_0$.

(43) $x_2 + x = x_3$.

(44) $x_4 + x_0 = x_3 + x_1$.

(45) $x_3 + x_0 = x_2 + x_1$.

(46) $x_3 + x = x_4$.

(47) $\delta(x_3, x_0) = x$.

(48) If $-x = -y$, then $\delta(x,z) = \delta(y,z)$.

(49) $\delta(x, -y) = \delta(y, -x)$.

(50) $\delta(x_3, x) = x_0$.

(51) $\delta(x_1 + x_3, x) = x_0$.

(52) $\delta(x_1 + x_2, x) = x_0$.

(53) $\delta(x_1 + x_3, x_0) = x$.

Let us consider G, x. The functor $\beta(x)$ yielding an element of G is defined as follows:

(Def. 22) $\beta(x) = -(x_1 + x_3) + x + -x_3$.

We now state three propositions:

(54) $\delta(\beta(x), x) = -x_3$.

(55) $\delta(\beta(x), x) = -(x_1 + x_3)$.

(56) There exist y, z such that $-(y + z) = -z$.

5. Proofs according to William McCune

One can prove the following two propositions:

(57) If for every z holds $--z = z$, then G is Huntington.

(58) If G has idempotent element, then G is Huntington.

Let us observe that TrivComplLat has idempotent element.

One can check that every Robbins join-associative join-commutative non empty complemented lattice structure which has idempotent element is Huntington.

One can prove the following two propositions:

(59) If there exist elements c, d of the carrier of G such that $c + d = c$, then G is Huntington.

(60) There exist y, z such that $y + z = z$.

One can verify that every join-associative join-commutative non empty complemented lattice structure which is Robbins is also Huntington.

Let L be a non empty ortholattice structure. We say that L is de Morgan if and only if:

(Def. 23) For all elements x, y of the carrier of L holds $x \sqcap y = (x^{\rm c} \sqcup y^{\rm c})^{\rm c}$.

Let L be a non empty complemented lattice structure. One can verify that CLatt L is de Morgan.

Next we state two propositions:

(61) Let L be a well-complemented join-commutative meet-commutative non empty ortholattice structure and x be an element of the carrier of L. Then $x + x^{\rm c} = \top_L$ and $x \sqcap x^{\rm c} = \perp_L$.

(62) For every bounded distributive well-complemented pre-ortholattice L holds $(\top_L)^{\rm c} = \perp_L$.

Let us observe that TrivOrtLat is de Morgan.

One can verify that there exists a pre-ortholattice which is strict, de Morgan, Boolean, Robbins, and Huntington.

Let us note that every non empty ortholattice structure which is join-associative, join-commutative, and de Morgan is also meet-commutative.

One can prove the following proposition

(63) For every Huntington de Morgan pre-ortholattice L holds $\bot_L^{\mathrm{C}} = \bot_L$.

One can verify that every well-complemented pre-ortholattice which is Boolean is also Huntington.

Let us note that every de Morgan pre-ortholattice which is Huntington is also Boolean.

One can verify that every pre-ortholattice which is Robbins and de Morgan is also Boolean and every well-complemented pre-ortholattice which is Boolean is also Robbins.

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