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Hierarchies and Classifications of Sets[1]

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Summary. This article is a continuation of [2] article. Further properties of classification of sets are proved. The notion of hierarchy of a set is introduced. Properties of partitions and hierarchies are shown. The main theorem says that for each hierarchy there exists a classification which the union is equal to the considered hierarchy.

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The terminology and notation used here have been introduced in the following articles: [7], [11], [6], [9], [4], [12], [5], [10], [8], [2], [3], and [1].

1. Tree and Classification of a Set

For simplicity, we follow the rules: A denotes a relational structure, X denotes a non empty set, P_1, P_2, P_3, Y, a, b, c, x denote sets, and S_1 denotes a subset of Y.

Let us consider A. We say that A has superior elements if and only if:

(Def. 1) There exists an element of A which is superior of the internal relation of A.

Let us consider A. We say that A has comparable down elements if and only if:

(Def. 2) For all elements x, y of A such that there exists an element z of A such that $z \leqslant x$ and $z \leqslant y$ holds $x \leqslant y$ or $y \leqslant x$.

The following proposition is true

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(1) For every set a holds $\langle \{\{a\}\}, \subseteq \rangle$ is non empty, reflexive, transitive, and antisymmetric and has superior elements and comparable down elements.

Let us observe that there exists a relational structure which is non empty, reflexive, transitive, antisymmetric, and strict and has superior elements and comparable down elements.

A tree is a poset with superior elements and comparable down elements.

Next we state four propositions:

(2) For every equivalence relation E_1 of X and for all sets x, y, z such that $z \in [x]_{(E_1)}$ and $z \in [y]_{(E_1)}$ holds $[x]_{(E_1)} = [y]_{(E_1)}$.

(3) For every partition P of X and for all sets x, y, z such that $x \in P$ and $y \in P$ and $z \in x$ and $z \in y$ holds $x = y$.

(4) For all sets C, x such that C is a classification of X and $x \in \bigcup C$ holds $x \subseteq X$.

(5) For every set C such that C is a strong classification of X holds $\langle \bigcup C, \subseteq \rangle$ is a tree.

2. The Hierarchy of a Set

Let us consider Y. We say that Y is hierarchic if and only if:

(Def. 3) For all sets x, y such that $x \in Y$ and $y \in Y$ holds $x \subseteq y$ or $y \subseteq x$ or x misses y.

One can verify that every set which is trivial is also hierarchic.

Let us note that there exists a set which is non trivial and hierarchic.

The following propositions are true:

(6) $\emptyset$ is hierarchic.

(7) $\{\emptyset\}$ is hierarchic.

Let us consider Y. A family of subsets of Y is said to be a hierarchy of Y if:

(Def. 4) It is hierarchic.

Let us consider Y. We say that Y is mutually-disjoint if and only if:

(Def. 5) For all sets x, y such that $x \in Y$ and $y \in Y$ and $x \neq y$ holds x misses y.

In the sequel H denotes a hierarchy of Y.

Let us consider Y. Observe that there exists a family of subsets of Y which is mutually-disjoint.

Next we state three propositions:

(8) $\emptyset$ is mutually-disjoint.

(9) $\{\emptyset\}$ is mutually-disjoint.

(10) $\{a\}$ is mutually-disjoint.

Let us consider Y and let F be a family of subsets of Y. We say that F is T_3 if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let A be a subset of Y. Suppose $A \in F$. Let x be an element of Y. If $x \notin A$, then there exists a subset B of Y such that $x \in B$ and $B \in F$ and A misses B.

We now state the proposition

(11) For every family F of subsets of Y such that $F = \emptyset$ holds F is T_3.

Let us consider Y. One can verify that there exists a hierarchy of Y which is covering and T_3.

Let us consider Y and let F be a family of subsets of Y. We say that F is lower-bounded if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let B be a set. Suppose $B \neq \emptyset$ and $B \subseteq F$ and for all a, b such that $a \in B$ and $b \in B$ holds $a \subseteq b$ or $b \subseteq a$. Then there exists c such that $c \in F$ and $c \subseteq \bigcap B$.

Next we state the proposition

(12) Let B be a mutually-disjoint family of subsets of Y. Suppose that for every set b such that $b \in B$ holds S_1 misses b and $Y \neq \emptyset$. Then $B \cup \{S_1\}$ is a mutually-disjoint family of subsets of Y and if $S_1 \neq \emptyset$, then $\bigcup(B \cup \{S_1\}) \neq \bigcup B$.

Let us consider Y and let F be a family of subsets of Y. We say that F has maximum elements if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let S be a subset of Y. Suppose $S \in F$. Then there exists a subset T of Y such that $S \subseteq T$ and $T \in F$ and for every subset V of Y such that $T \subseteq V$ and $V \in F$ holds $V = Y$.

3. Some Properties of Partitions, Hierarchies and Classifications of Sets

The following propositions are true:

(13) For every covering hierarchy H of Y such that H has maximum elements there exists a partition P of Y such that $P \subseteq H$.

(14) Let H be a covering hierarchy of Y and B be a mutually-disjoint family of subsets of Y. Suppose $B \subseteq H$ and for every mutually-disjoint family C of subsets of Y such that $C \subseteq H$ and $\bigcup B \subseteq \bigcup C$ holds $B = C$. Then B is a partition of Y.

(15) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y and B be a mutually-disjoint family of subsets of Y. Suppose that

(i) $A \in B$,

(ii) $B \subseteq H$, and

(iii) for every mutually-disjoint family C of subsets of Y such that $A \in C$ and $C \subseteq H$ and $\bigcup B \subseteq \bigcup C$ holds $\bigcup B = \bigcup C$.

Then B is a partition of Y.

(16) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y and B be a mutually-disjoint family of subsets of Y. Suppose $A \in B$ and $B \subseteq H$ and for every mutually-disjoint family C of subsets of Y such that $A \in C$ and $C \subseteq H$ and $B \subseteq C$ holds $B = C$. Then B is a partition of Y.

(17) Let H be a covering T_3 hierarchy of Y. Suppose H is lower-bounded and $\emptyset \notin H$. Let A be a subset of Y. If $A \in H$, then there exists a partition P of Y such that $A \in P$ and $P \subseteq H$.

(18) Let h be a non empty set, P_4 be a partition of X, and h_1 be a set. Suppose $h_1 \in P_4$ and $h \subseteq h_1$. Let P_6 be a partition of X. Suppose $h \in P_6$ and for every x such that $x \in P_6$ holds $x \subseteq h_1$ or $h_1 \subseteq x$ or h_1 misses x. Let P_5 be a set. Suppose that for every a holds $a \in P_5$ iff $a \in P_6$ and $a \subseteq h_1$. Then $P_5 \cup (P_4 \setminus \{h_1\})$ is a partition of X and $P_5 \cup (P_4 \setminus \{h_1\})$ is finer than P_4.

(19) Let h be a non empty set. Suppose $h \subseteq X$. Let P_8 be a partition of X. Suppose there exists a set h_2 such that $h_2 \in P_8$ and $h_2 \subseteq h$ and for every x such that $x \in P_8$ holds $x \subseteq h$ or $h \subseteq x$ or h misses x. Let P_7 be a set. Suppose that for every x holds $x \in P_7$ iff $x \in P_8$ and x misses h. Then

(i) $P_7 \cup \{h\}$ is a partition of X,

(ii) P_8 is finer than $P_7 \cup \{h\}$, and

(iii) for every partition P_4 of X such that P_8 is finer than P_4 and for every set h_1 such that $h_1 \in P_4$ and $h \subseteq h_1$ holds $P_7 \cup \{h\}$ is finer than P_4.

(20) Let H be a covering T_3 hierarchy of X. Suppose that

(i) H is lower-bounded,

(ii) $\emptyset \notin H$, and

(iii) for every set C_1 such that $C_1 \neq \emptyset$ and $C_1 \subseteq \mathrm{PARTITIONS}(X)$ and for all sets P_9, P_{10} such that $P_9 \in C_1$ and $P_{10} \in C_1$ holds P_9 is finer than P_{10} or P_{10} is finer than P_9 there exist P_1, P_2 such that $P_1 \in C_1$ and $P_2 \in C_1$ and for every P_3 such that $P_3 \in C_1$ holds P_3 is finer than P_2 and P_1 is finer than P_3.

Then there exists a classification C of X such that $\bigcup C = H$.

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